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**Real Analysis** **Qualifying Exam** **August 2014**

*Instructions:* Please hand in all of the 8 following problems. Start each problem on a new page, number the pages, and put only your Banner identification number on each page. Clear and concise answers with good justification will improve your score.

1. Let $\mathcal{X}$ be the space of real-valued sequences whose terms form an absolutely convergent series, more precisely,

$$\mathcal{X} := \left\{ (a_n)_{n=0}^{\infty} : \; a_n \in \mathbb{R} \;\text{ and }\; \sum_{n=0}^{\infty} |a_n| < \infty \right\}.$$

Define $d : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ as follows

$$d(A, B) := \sum_{n=0}^{\infty} |a_n - b_n|.$$

where $A, B$ denote the sequences $(a_n)_{n=0}^{\infty}$, $(b_n)_{n=0}^{\infty}$ respectively.

   (a) Show that $d$ is a metric on $\mathcal{X}$.

   (b) For each $j \in \mathbb{N}$, let $E^{(j)} := (e_n^{(j)})_{n=0}^{\infty}$ be a sequence in $\mathcal{X}$ where $e_n^{(j)} = 1$ if $n = j$ and $e_n^{(j)} = 0$ if $n \neq j$. Show that $\mathcal{S} := \{E^{(j)} : j \in \mathbb{N}\}$ is a closed and bounded subset of $\mathcal{X}$ with respect to the $\ell^1$ metric.

   (c) Is $\mathcal{S}$ a compact subset of $\mathcal{X}$ with respect to the metric $d$?

2. Assume $g : (a, c) \to \mathbb{R}$ and is uniformly continuous on the subinterval $(a, b]$ and on the subinterval $[b, c)$ where $a < b < c$. In other words, the restriction of $g$ to $(a, b]$ and the restriction of $g$ to $[b, c)$ both define uniformly continuous functions. Prove that $g$ is uniformly continuous on the full interval $(a, c)$.

3. Suppose $f$ is a real valued function defined in a neighborhood of a point $x_0 \in \mathbb{R}$ and that $f'$ exists in that same neighborhood. If $f''(x_0)$ exists, show that

$$\lim_{h \to 0} \frac{f(x_0 + h) + f(x_0 - h) - 2f(x_0)}{h^2} = f''(x_0).$$

Then show by example that the limit may exist even if $f''(x_0)$ does not.

4. Suppose $f_n, g_n : \mathbb{R} \to \mathbb{R}$ are two sequences of functions converging uniformly on $\mathbb{R}$ to functions $f, g$ respectively.

   (a) Show that if both sequences are uniformly bounded, then the products $f_n g_n$ converge uniformly to $fg$.

   (b) Show by example that the conclusion in part (a) may fail to hold if the sequences are not assumed to be uniformly bounded.

5. Consider the function $f : [0, 1] \to \mathbb{R}$

$$f(x) = \begin{cases} \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Show that $f$ is Riemann integrable on $[0, 1]$.

6. Suppose $F : \mathbb{R}^3 \to \mathbb{R}^2$ is a continuously differentiable function given as $F(\vec{x}) = (f_1(\vec{x}), f_2(\vec{x}))$ for some scalar valued functions $f_1$, $f_2$ and $\vec{x} = (x_1, x_2, x_3)$. Suppose further that $\vec{a} \in \mathbb{R}^3$ is such that $F'(\vec{a})$ has rank 2. Prove that there exists a function $f_3 : \mathbb{R}^3 \to \mathbb{R}$ such that

$$\Phi(x) = (f_1(\vec{x}), f_2(\vec{x}), f_3(\vec{x}))$$

as a function from $\mathbb{R}^3$ to itself has a continuous inverse near $\vec{a}$.

7. Find $\iint_E \cos(3x^2 + y^2)\, dxdy$ where $E$ is the set of points

$$E := \left\{ (x, y) \in \mathbb{R}^2 : x^2 + \frac{y^2}{3} \leq 1 \right\}.$$

8. Let $f : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}$ be defined by

$$f(x, y, z) = \frac{1}{\sqrt{x^2 + y^2 + z^2}}.$$

(a) Show that $f$ is *harmonic* on $\mathbb{R}^3 \setminus \{0\}$, that is,

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \operatorname{div}(\nabla f) = 0$$

at every point $(x, y, z) \neq 0$.

(b) Let $\iint_{\mathbb{S}_r} \nabla f \cdot d\mathbf{S}$ denote the surface integral of the vector field $\nabla f$ over the sphere of radius $r$, oriented by outward pointing normals. Show that $\iint_{\mathbb{S}_r} \nabla f \cdot d\mathbf{S}$ is independent of $r > 0$, that is, if $0 < r_1 < r_2 < \infty$ then

$$\iint_{\mathbb{S}_{r_1}} \nabla f \cdot d\mathbf{S} = \iint_{\mathbb{S}_{r_2}} \nabla f \cdot d\mathbf{S}$$