COMPLEX ANALYSIS QUALIFYING EXAMINATION

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DIRECTIONS: You are trying to convince the reader that you know what your are doing, so you should give clear, concise and complete answers explaining your work. Note that holomorphic is synonymous with analytic. **Do any 7 of the following 10 problems.**

1. Let $z = x + iy$ and $f(z) = \sqrt[2]{|xy|}$. Prove that $f(z)$ satisfies the Cauchy-Riemann equations at $z = 0$ but that $f(z)$ is not complex differentiable at $z = 0$.

2. Compute the radius of convergence of the following:

a) $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{n^{2n}} z^n.$

b) $\displaystyle\sum_{n=0}^{\infty} (n + a^n) z^n,$ where $a \in \mathbb{C}$.

c) The Taylor series around zero for the function $z \cot z$.

3. Let $T(z) = \dfrac{az+b}{cz+d}$ be a linear fractional transformation with $a, b, c, d \in \mathbb{C}$. Suppose also that T has two fixed points in $\mathbb{C}$. Prove that the product of the derivatives of T at the two fixed points equals one, i.e. that $T'(z_1)T'(z_2) = 1$, where z_1, z_2 are the two fixed points.

4. Prove Goursat's Lemma: If $f(z)$ is holomorphic in a simply connected region $\Omega \subset \mathbb{C}$ then for the boundary $\partial\Delta$ of every triangle $\Delta \subset \Omega$ one has $\displaystyle\int_{\partial\Delta} f = 0$.

5. Prove the Schwarz Lemma: If $f(z)$ is a holomorphic function in the unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$ satisfying $f(0) = 0$ and $|f(z)| \leq 1$ for all $z \in D$, then $|f(z)| \leq |z|$ for all $z \in D$.

6. Use the method of contour integration and the calculus of residues to evaluate the integral $\displaystyle\int_0^{\infty} \frac{x^p}{1+x^2} dx$ where $-1 < p < 1$. Draw the relevant contour and justify all steps.

7. Determine the three Laurent series around 0 of the function $f(z) = \dfrac{1}{(z-1)(z-2)}$ in the three regions $|z| < 1, 1 < |z| < 2$, and $|z| > 2$, respectively.

8. Find the number of zeroes of the equation $e^z - 4z^n + 1 = 0$ in the unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$.

9. Do parts a), b) and c), below.

a) State the Mittag-Leffler Theorem on the existence of a meromorphic function with an (infinite) discrete set of poles with prescribed principal parts.

b) State the Weierstrass Theorem on the existence of an entire function with an (infinite) discrete set of zeros with prescribed locations and orders.

c) Prove either one of the theorems in part a or b. Indicate clearly which theorem you are proving.

10. Suppose $f(z)$ is holomorphic in a neighborhood U of the origin 0 and that $f'(0) \neq 0$. Prove that there are a disc $B \subset U$, a positive integer n, and a holomorphic function $g(z)$ such that $f(z) = f(0) + (g(z))^n$ in B.