# Complex Variables Master's - Qualifying Examination

## August 1995

1. Let $\{a_n\}$ be a sequence of complex numbers. Assume that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ converges for all $z$ satisfying $|z| \leq r$. Prove that if $|a_1| > \sum_{n=2}^{\infty} n|a_n|r^{n-1}$, then $f$ is an injective function on the disc $|z| \leq r$.

2. Let $f(z) = \frac{1}{z-1} + \frac{1}{(z-2)^2}$. Expand $f(z)$ in a

   i) Taylor series in $|z| < 1$

   ii) Laurent series in $1 < |z| < 2$.

3. A complex-valued function $f = U + iV$ is said to be harmonic on a domain $D \subset \mathbb{C}$ if $U$ and $V$ are harmonic on $D$. Show that $f$ is holomorphic on $D$ if and only if both $f$ and $zf$ are harmonic on $D$.

4. (a) Show $\sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}$ is meromorphic in the complex plane $\mathbb{C}$

   (b) Argue that $\frac{\pi^2}{(\sin \pi z)^2} - \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}$ is holomorphic in $\mathbb{C}$

   (c) Show that this holomorphic function is 0, i.e. $\frac{\pi^2}{(\sin \pi z)^2} = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}$

5. Let $f : U \longrightarrow \mathbb{C}$ be a holomorphic function defined on an open subset $U$ of $\mathbb{C}$. Let $R$ be a closed rectangle contained in $U$ (assume that the sides of $R$ are parallel to the lines $\mathrm{Re}z = 0$ and $\mathrm{Im}z = 0$). Give a complete proof of the equality

$$\int_{\partial R} f(z)dz = 0$$

6. Let $p_n(z) = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!}$. Prove that for every $R > 0$ there exists a positive integer $n(R)$ such that all roots of $p_n(z) = 0$ for $n \geq n(R)$ belong to the set $\{z \in \mathbb{C} \,\big|\, |z| > R\}$.

7. Assume that $a, b, c$ are real numbers satisfying $ac - b^2 > 0$. Prove using residues

$$\int_{-\infty}^{\infty} \frac{dx}{ax^2 + 2bx + c} = \frac{\pi}{\sqrt{ac - b^2}}.$$

8. Let $\{a_n\}$ and $\{b_n\}$ be sequences of complex numbers. Assume that $\{a_n\}$ has no accumulation point. Prove that there exists a holomorphic function $f : \mathbb{C} \longrightarrow \mathbb{C}$ such that $f(a_n) = b_n$.