Complex Analysis Qualifying Examination

August 1996

Directions:

1. You are trying to convince the reader that you know what you are doing. To that end we suggest your presentation be *clear, concise* and *complete.*
2. Solve any eight problems.
3. Start each question on a new sheet of paper. Write only on one side of each sheet of paper. Number the pages.

Terminology:

1. A *domain* is a non-empty open connected set in the complex plane.
2. A sequence or series of functions is said to be *locally uniformly convergent* on a domain Ω if it converges uniformly on every compact subset of Ω.

Notation:

1. $D(a, r) = \{z \in \mathbb{C}\,; |z-a| < r\}$ $(r > 0)$.
2. $f(z) \in H(\Omega)$ means that $f(z)$ is analytic on the domain Ω.

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1. Suppose $f(z) = \sum_{n=0}^{\infty} a_n(z-c)^n$ has the property that the series $\sum_{n=0}^{\infty} f^{(n)}(c)$ converges. Show that $f(z)$ is an entire function.

2. Show that an entire function that takes real values on the real axis and purely imaginary values on the imaginary axis must be an odd function:

$$f(-z) = -f(z) \quad \text{for all } z \in \mathbb{C}.$$

3. Let $f(z)$ be analytic in the unit disc. Define $g(w) = f(z)$ where $w = Tz$ is a Möbius transformation mapping the unit disc conformally onto itself. Show that

$$(1-|w|^2)\left|\frac{dg}{dw}\right| = (1-|z|^2)\left|\frac{df}{dz}\right|.$$

4. Let $f_1(z), f_2(z), \cdots, f_n(z) \in H(\Omega)$, and

$$\varphi(z) = |f_1(z)|^2 + |f_2(z)|^2 + \cdots + |f_n(z)|^2.$$

   (a) Show that $\varphi(z)$ is harmonic on the domain Ω only if all the functions $f_k(z)$ $(k = 1, 2, \cdots, n)$ reduce to constant functions.

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(b) Show that $\varphi(z)$ has no local maximum in Ω unless all the functions $f_k(z)$ $(k = 1, 2, \cdots, n)$ reduce to constant functions.

5. Find all entire functions that satisfies the Lipschitz condition on $\mathbf{C}$. A function $f(z)$ is said to satisfy the *Lipschitz condition* on $\mathbf{C}$ if there exists a positive constant M such that

$$|f(z_1) - f(z_2)| \leq M \cdot |z_1 - z_2| \quad \text{for all } z_1, z_2 \in \mathbf{C}.$$

6. Find the *Fourier transform* of the function $f(x) = e^{-x^2/2}$; i.e., find the function $\hat{f}(t)$ defined for all $t \in \mathbf{R}$ by

$$\hat{f}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} \cdot e^{-itx} dx.$$

7. Show that $z^5 - 15z + 1 = 0$ has one root in the disc $|z| < \frac{1}{8}$ and four roots in the annulus $\frac{3}{2} < |z| < 2$.

8. If $f(z)$ is analytic in the unit disc and $f(0) = 0$, show that

$$f(z) + f(z^2) + \cdots + f(z^n) + \cdots$$

converges locally uniformly to an analytic function in the unit disc.

9. Construct an entire function $f(z)$ such that

$$f(n) = n! \quad (n = 0, 1, 2, \cdots).$$

10. Find the conformal mapping $w = f(z)$ of a convex lens $D(\sqrt{3}, 2) \cap D(-\sqrt{3}, 2)$ onto the unit disc satisfying $f(0) = 0$, $f'(0) > 0$.