Complex Analysis Qualifying Examination

August 1998 SS # : ____________________

Directions:

1. You are trying to convince the reader that you know what you are doing. To that end we suggest your presentation be *clear, consise* and *complete.*

2. Start each question on a new sheet of paper. Write only on one side of each sheet of paper. Number the pages and *write your name* in each page.

Questions:

Part A: Answer *four* out of the five questions below.

1. Expand $f(z) = \dfrac{z+6}{z^2-2z-3}$ in

   (a) Taylor series around $z = 0$.

   (b) Laurent series in the annulus $1 < |z| < 3$.

   (c) Laurent series in the region $3 < |z| < \infty$.

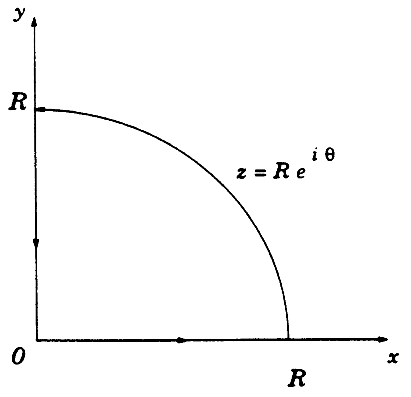
2. Find the bilinear transformation which takes $z_1 = 0$, $z_2 = 1$, and $z_3 = 2$ to $w_1 = 2i$, $w_2 = -2$, and $w_3 = -2i$.

3. Evaluate

   (a) $\displaystyle\int_{|z|=2} \frac{z+6}{z^2-2z-3}\,dz$

   (b) $\displaystyle\int_0^{\pi} \frac{d\theta}{6-3\cos\theta}$

4. Consider $I = \displaystyle\int_\gamma \frac{z^2\,dz}{1+z^4}$, where γ is the contour shown below



   (a) Evaluate I when $R < 1$.
   (b) Evaluate I when $R > 1$.
   (c) Discuss the results obtained when $R \to \infty$.

5. (a) State and prove Liouville's theorem.
   (b) Deduce the fundamental theorem of algebra from part (a).

Part B: Answer *four* out of the six questions below.

1. Let $f(z) = \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}$

   (a) Show that $f(z)$ is analytic in the complex plane minus the non-zero integers.

   (b) Show that if $f(z) = 0$, then z is real.

2. Recall that a polynomial $p(z)$ for which $p(0) > 0$ and which has only negative real zeros has positive coefficients. Now consider the function

$$f(z) = \frac{1}{z\Gamma(z)}$$

   which has only negative real zeros ($\Gamma(z)$ is the Gamma function). Are all the Taylor coefficients of $f(z)$ positive? Explain.

3. Let $w = f(z) = \sum_{0}^{\infty} a_n z^n$ be analytic and univalent in a disc D of radius r, centered at $z = 0$. Show that the area A of the image $f(D)$ is given by

$$A = \pi \sum_{n=1}^{\infty} n|a_n|^2 r^{2n}$$

4. (a) State and prove a form of the maximum principle.
   (b) State Schwarz's lemma and give a proof.

5. (a) Explain why there is an entire function $f(z)$ such that

$$e^{f(z)} = \frac{e^{2z} + 1}{\cos(iz)}.$$

   (b) Find an entire function $f(z)$ satisfying the above relation.

6. (a) State and prove Rouche's theorem.
   (b) Use Rouche's theorem to show that there is $\epsilon_0 > 0$ so that for $0 < \epsilon < \epsilon_0$ the equation $z^3 - \epsilon z - 1 = 0$ has three distinct roots.