Complex Analysis Qualifying Exam

January 1999

Do the following 8 problems. Show all your work and explain all steps in a proof or derivation.

1. State the Cauchy-Goursat Integral Theorem and give an outline of its proof.

2. Classify the singularities at $z = 0$ of the following functions $f(z)$ and find their residue.

   a) $f(z) = \frac{1}{z}$.

   b) $f(z) = z\cos(\frac{1}{z})$.

   c) $f(z) = z^{-3}\csc(z^2)$.

3. Consider the following meromorphic functions.

   a) Expand $f(z) = \frac{1}{2z - z^2}$ in a power series about $z = 1$.

   b) Find a Laurent expansion of $f(z) = \frac{1}{z} + \frac{1}{z+2} + \frac{1}{(z-1)^2}$ which is valid in the annulus $1 < |z| < 2$.

4. Show that $\tan z = z$ has no complex solutions of the form $z = x + iy$ with $x \neq 0, y \neq 0$.

5. Use the theory of residues to evaluate the integral $\int_0^\infty \frac{\sqrt{x}dx}{x^2+4}$.

6. State Rouche's theorem and use it to determine how many roots of the polynomial $z^4 + 5z + 3$ lie inside:

   a) the unit disc.

   b) the annulus $1 < |z| < 2$.

7. Prove the Casorati-Weierstrass Theorem: Let z_0 be an isolated essential singularity of a function $f(z)$. Then the image of a punctured neighborhood $U\backslash\{z_0\}$ under f is dense in the complex plane $\mathbb{C}$.

8. State the Mittag-Leffler Theorem and use it to prove that $\frac{\pi^2}{\sin^2 \pi z} = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}$.