Complex Analysis Qualifying Exam

August 1999

Do the following 8 problems. Show all your work and explain all steps in a proof or derivation.

1. Let f be analytic on $\mathbb{C}$ and real-valued on the circle $|z| = 1$. Show tht f is a constant on $\mathbb{C}$.

2. Classify the singularities at $z = 0$ of the following functions $f(z)$ (including the point at ∞).

   a) $f(z) = \dfrac{\sin^2 z}{z^4}$.

   b) $f(z) = \sin(\dfrac{1}{z}) + \dfrac{1}{z^2(z-1)}$.

   c) $f(z) = \csc z - \dfrac{1}{z}$.

3. Show the map $f(z) = \dfrac{z-i}{z+i}$ is a bijection of the upper half plane $H = \{z \in \mathbb{C} | \text{Im } z > 0\}$ onto the unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$.

4. Let f be a continuous map of a connected open subset $U \subset \mathbb{C}$ into $\mathbb{C}$.

   a) Show that f has a primitive on U if and only if $\int_\gamma f(z)dz = 0$ for every simple closed curve γ contained in U.

   b) Consider $f(z) = \frac{1}{z}$ on the punctured unit disc $D(0,1) - \{0\}$. Does f have a primitive on $D(0,1) - \{0\}$? Explain.

5. Use the theory of residues to evaluate the integral $\displaystyle\int_0^\infty \frac{\ln x \, dx}{x^2 + a^2}$.

6. State the Argument Principle and use it to prove the Open Mapping Theorem: Let f be analytic on some region Ω. Then the image $f(U)$ is open in $\mathbb{C}$ for every open set $U \subset \Omega$. Hint. Apply the Argument Principle to the function $f(z) - w$.

7. Use the Casorati-Weierstrass Theorem to prove that if the composition $f \circ g$ of two entire functions f and g is a polynomial, then both f and g are polynomials.

8. Let f be a bounded analytic function on the disc $D(0, R)$. Suppose that f also satisfies $f^{(i)}(0) = 0$ for all $i = 0, \cdots, k$. Show that f satisfies the inequality $|f(z)| \le \dfrac{M}{R^{k+1}}|z|^{k+1}$ on $D(0, R)$ where $M = \sup_{z \in D(0,R)} |f(z)|$.