Complex Variables
Master's Examination

Spring 2000

Instructions: There are nine (9) questions on this examination, and each question is worth 25 points. Work any 8 problems. A maximum score of 200 points is possible.

1. Find a conformal mapping of the strip $0 < \Re z < 1$ onto the unit disk in such a way that $z = 1/2$ goes to $w = 0$ and $z = \infty$ goes to $w = 1$.

2. According to the Weirstrass factorization theorem, $f(z) = \cos\sqrt{z}$ can be written as an infinite product
$$f(z) = Ce^{g(z)}z^m \prod_1^\infty \left(1 - \frac{z}{a_n}\right) e^{h_n(z)} \quad ,$$
where $g(z)$ is entire and
$$h_n = \begin{cases} 0, & k = 0 \\ \frac{z}{a_n} + \frac{1}{2}\left(\frac{z}{a_n}\right)^2 + \cdots + \frac{1}{k}\left(\frac{z}{a_n}\right)^k \;, & k \in \mathbb{Z}^+ \end{cases}$$
Determine a_n, k and m

3. According to the Mittag-Leffler theorem, the meromorphic function
$$f(z) = \frac{\pi^2}{\sin^2 \pi z} \quad ,$$
can be expressed by the infinite series
$$f(z) = \sum_k \left[P_k\left(\frac{1}{z - b_k}\right) - p_k(z)\right] + g(z) \quad ,$$
where $P_k(z)$, $p_k(z)$ are appropriately chosen polynomials, b_k are appropriately chosen complex numbers and $g(z)$ is analytic in the entire complex plane. Determine b_k, P_k and p_k.

4. For $a, b > 0$, evaluate the integral
$$\int_0^\infty \frac{\cos ax}{x^2 + b^2} dx \ .$$
Carefully justify any estimate you make.

5. Evaluate the integral
$$\int_0^\infty \frac{x^p}{1 + x^2} dx \ ,$$
with $-1 < p < 1$ by contour integration. As in the previous problem, carefully justify all your estimates.

6. Let $B(0;1) = \{z \in \mathbb{C} | \ |z| < 1\}$ be the unit disk. If $a > e$, and n is a positive integer, prove that the equation $e^z = az^n$ has n distinct roots in $B(0;1)$ (counted with multiplicity).

7. Let $\Omega \subset \mathbb{C}$ be a simply connected region, and $u : \Omega \to \mathbb{R}$ be a harmonic function. Prove that there exists $v : \Omega \to \mathbb{R}$ such that $u + iv$ is analytic on Ω.
Hint: Consider
$$g(z) = \frac{\partial u}{\partial x} + i\left(-\frac{\partial u}{\partial y}\right) \quad .$$

8. Assume that $f(z)$ is analytic on $\mathbb{C} \setminus \{0\}$ and
$$|f(z)| \leq |z|^2 + \frac{1}{|z|^{1/2}}$$
for all $z \in \mathbb{C} \setminus \{0\}$. Prove that f is a polynomial of degree at most 2.

9. Let $f(z)$ be continuous on the closed right half-plane $\tilde{\mathcal{H}} = \{z \in \mathbb{C} | \Re z \geq 0\}$ and analytic on the open right half-plane $\mathcal{H} = \{z \in \mathbb{C} | \Re z > 0\}$. Suppose there exist constants C, $M \in \mathbb{R}$ and a positive integer n such that
(a) $|f(iy)| \leq M$ for all $y \in \mathbb{R}$,
(b) $|f(z)| \leq C\,(1 + |z|^n)$ for all $z \in \mathcal{H}$.
Prove that $|f(z)| \leq M$ for all $z \in \mathcal{H}$.
Hint: For $\epsilon > 0$, consider
$$f_\epsilon(z) := \frac{f(z)}{(1 + \epsilon z)^{n+1}}$$
and apply the maximum principle.