Complex Analysis Qualifying Examination

August 2000 SS # : ____________________

Directions:

Do the following 8 problems. You may choose to answer the problems in any order. Please start each question on a new sheet of paper. Write only on one side of each sheet of paper. Number the pages and *write your SS #* in each page. Please show all your work and explain all steps in a proof or derivation.

Questions:

1. (a) Find the linear fractional transformation $w = f(z)$ for which $f(0) = 0$, $f(2) = 4$, $f(\mathrm{i}) = 1 - \mathrm{i}$, and its inverse.

   (b) Obviously $z = 0$ is a fixed point; are there any other fixed points?

   (c) Describe the image of the region $1 \leq y$ in the w-plane.

2. Classify the singularities (including the point at ∞) and find the residues for

   a) $f(z) = \sin\left(\dfrac{1}{z}\right)$ b) $f(z) = \dfrac{\sin(z^2)}{z^7}$ c) $f(z) = \dfrac{1}{z^2}\cot z$.

3. Expand the function $f(z) = \dfrac{z^3 + 2z - 4}{z}$ in power series around $z = 1$ and give its radius of convergence.

4. Evaluate the real integral (and justify all steps)

$$\int_0^\infty \frac{x^2}{(x^2+1)^2}\,dx$$

5. (a) Show that $u(x, y) = x^3 - 3xy^2 + y^2 - x^2$ is harmonic in the entire plane.

   (b) Find a harmonic conjugate $v(x, y)$.

   (c) Give explicitely an analytic function $w = f(z)$ with $u = \mathrm{Re} f$ and $v = \mathrm{Im} f$.

6. State and prove the Cauchy integral formula.

7. Assume that $w = f(z) = u(z) + \mathrm{i}v(z)$ is an analytic function mapping a domain D in the z-plane onto a domain D' in the w-plane. If $\phi(u, v)$ is a harmonic function in D', show that the function

$$\Phi(x, y) = \phi(u(x, y), v(x, y))$$

   is harmonic in D.

8. (a) State Rouche's theorem.

   (b) Find the number of zeros of $f(z) = 2z^5 + 7z^3 + z^2 - 3$ in the annulus $1 < |z| < 2$.