Complex Analysis Qualifying Examination

January 2004 SS#: ____________

Instructions: Please do **any eight** out of the nine problems listed below. You may choose to answer the problems in any order. However, to help us in grading your exam please make sure to:
i. Start each question on a new sheet of paper.
ii. Write **only on one side** of each sheet of paper.
iii. Number each page and write your SS# in each page.
Good luck!

1. If $f(z)$ is analytic in the unit disc $D = \{z \in C \mid |z| < 1\}$, prove that there is an analytic function $F(z)$ in D with $F'(z) = f(z)$.
   You can quote the Cauchy-Goursat theorem.

2. (a) Find the Möbius transformation that sends the points $z = 0, \infty, i$ into $w = -1, 1, i$ respectively.
   (b) Find the image of the first quadrant under the transformation found in part (a).

3. Find the Laurent expansion of $f(z) = (1 - z^2)e^{1/z}$ around $z = 0$. Determine its annulus of convergence and the residue of $f(z)$ at $z = 0$.

4. Compute the integral $I = \dfrac{1}{2\pi} \displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x(x^2+1)}\, dx$. Carefully justify all your steps.

5. Assume that $f(z)$ is an entire function with $\displaystyle\lim_{|z|\to\infty} \frac{f(z)}{z^2} = 0$. Prove that $f(z)$ must be linear, that is $f(z) = a + bz$, with $a, b \in C$. Please provide all the details.

6. Assume that $\sqrt{z}$ is given in terms of its principal branch, $0 \leq \arg(z) < 2\pi$. Find the image of the half-disc $D^+ = \{(x, y) \in C \mid x^2 + y^2 < 1,\ 0 < y < \infty\}$ under the map $w = \dfrac{1}{2}\left(\sqrt{z} + \dfrac{1}{\sqrt{z}}\right)$.

7. Show that if $f(z)$ is an entire function that never vanishes, then there is an entire function $g(z)$ so that $f(z) = e^{g(z)}$. Please provide all the details.

8. (a) Show that $u(x, y) = (x + 1)y$ is harmonic in the entire plane.
   (b) Find a harmonic conjugate $v(x, y)$ of $u(x, y)$.
   (c) Give explicitly an analytic function $w = f(z)$ with $u = \mathrm{Re} f$ and $v = \mathrm{Im} f$.

9. Prove the following version of the Schwarz reflection principle: if $f(z)$ is analytic in the right plane $\mathrm{Re} z > 0$, continuous in $\mathrm{Re} z \geq 0$, and $f(z)$ is purely imaginary on the imaginary axis, $\mathrm{Re} f(iy) = 0$, then $f(z)$ can be extended to an analytic function on the entire plane. Please provide all the details.