Complex Analysis Qualifying Exam

August 2004

Directions: Do all of the following problems. Show all of your work, and justify all of your calculations.

1. Evaluate

(a) $\displaystyle\int_0^{\infty} \mathrm{e}^{-x^2} \cos \lambda x \, \mathrm{d}x$

(b) $\displaystyle\int_0^{2\pi} \frac{\mathrm{d}\theta}{1+\cos^2\theta}$

It may be helpful to know that

$$\int_{-\infty}^{+\infty} \mathrm{e}^{-x^2}\,\mathrm{d}x = \sqrt{\pi}.$$

2. Classify *all* of the singularities and find the associated residues for each of the following functions:

(a) $z\cos 2z$

(b) $\dfrac{z^3 - z^2 + 2}{z-1}$.

3. Let $f : \mathbb{C} \mapsto \mathbb{C}$ be entire, and set $g(z) := f(1/z)$. Prove that f is a polynomial if and only if $g(z)$ has a pole at $z = 0$.

4. Let $U \subset \mathbb{C}$ be open, and let $f \in C^0(U)$. For each $r \in \mathbb{R}^+$ set $D(0,r) := \{z \in \mathbb{C} : |z| < r\}$. Suppose that for every $\overline{D}(z_0, r) \subset U$ and for all $z \in \overline{D}(z_0, r)$ one has that

$$f(z) = \oint_{\partial D(z_0,r)} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta$$

(the curve $\partial D(z_0, r)$ is followed in the counterclockwise direction). Prove that f is analytic.

5. Set

$$f(z) := \ln(z + (z^2-1)^{1/2}).$$

The branch cut for $(z^2-1)^{1/2}$ is to be on the axis $\operatorname{Im} z = 0$ with $-1 \le \operatorname{Re} z \le 1$.

(a) Show that $(z^2-1)^{1/2}$ is analytic at all values of $z \in \mathbb{C}$ not on the branch cut.

(b) For the transformation $w := z + (z^2-1)^{1/2}$ determine the branch cut for $f(w)$.

6. Let $D := D(0,1)$, and suppose that $f : D \mapsto \mathbb{C}$ is analytic, and further suppose that f is continuous on $\overline{D}$. Assume that $f(z) \neq 0$ for all $z \in \overline{D}$, and that $|f(z)| = 1$ for $z \in \partial D$. Show that $f(z) = \mathrm{e}^{\mathrm{i}\theta}$ for some $\theta \in [0, 2\pi)$.

7. For each $n \in \mathbb{N}$ set

$$f_n(z) := \sum_{j=1}^{n} \frac{z^{-j}}{j!}.$$

For a given $\rho > 0$, show that there is an $N(\rho)$ such that if $n > N(\rho)$, then all of the zeros of $f_n(z)$ lie within $D(0, \rho)$.

8. Let

$$f(z) := \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}.$$

(a) Compute the order of $f(z)$.

(b) Write the Hadamard product expansion of $f(z)$.