Complex Analysis Qualifying Exam

August 2005

Directions: Do all of the following problems. Show all of your work, and justify all of your calculations.

1. Classify *all* of the singularities and find the associated residues for each of the following functions:

(a) $\dfrac{(z+3)^2}{z}$

(b) $\dfrac{\mathrm{e}^{-z}}{(z-1)(z+2)^2}$.

2. Consider

$$f(z) := \sqrt{(z-x_1)(z-x_2)},$$

where $x_1, x_2 \in \mathbb{R}$ with $x_1 < x_2$. Upon writing

$$z - x_j = r_j \mathrm{e}^{\mathrm{i}\theta_j},$$

if one supposes that

$$0 \le \theta_1 < 2\pi, \quad -\pi \le \theta_2 < \pi,$$

determine the branch points and branch cuts for $f(z)$.

3. Show that

$$\int_{-\infty}^{+\infty} \frac{\cos x - \cos a}{x^2 - a^2}\,\mathrm{d}x = -\pi\frac{\sin a}{a}, \quad a \in \mathbb{R}^+.$$

4. Recall that a linear fractional transformation (LFT) is of the form

$$\ell(z) = \frac{az+b}{cz+d}, \quad ad - bc \ne 0.$$

(a) Find a LFT which maps the upper half-plane onto itself and which satisfies $\ell(0) = 1$, $\ell(\mathrm{i}) = 2\mathrm{i}$.

(b) Suppose that an LFT $\ell(z)$ has two distinct and finite fixed points α and β. Show that there is a constant $C \in \mathbb{C}$ such that

$$\frac{\ell(z)-\alpha}{\ell(z)-\beta} = C\frac{z-\alpha}{z-\beta}.$$

5. Let $f : D(P,r)\backslash\{P\} \mapsto \mathbb{C}$ be holomorphic, and suppose that f has an essential singularity at $z = P$. Show that there exists a sequence $\{z_j\} \subset D(P,r)\backslash\{P\}$ with $z_j \to P$ such that for each $j \in \mathbb{N}$,

$$|(z_j - P)^j f(z_j)| \ge j.$$

6. State some version of Rouche's theorem, and then use it to show that all of the zeros for

$$f(z) := z^8 - 4z^3 + 10$$

lie in the annulus $D(0,2)\backslash\overline{D}(0,1)$.

7. Set

$$h(z) := \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) \mathrm{e}^{-z/n}.$$

(a) Show that $h(z)$ is an entire function.

(b) The gamma function, $\Gamma(z)$, is nonzero and has simple poles at $z = 0, -1, -2, \ldots$. Show that there exists an entire function $g(z)$ such that

$$\frac{1}{\Gamma(z)} = z\mathrm{e}^{g(z)} h(z).$$

8. For each $n \in \mathbb{N}$ consider the polynomial

$$P_n(z) := 1 + z + z^2 + \cdots + z^n.$$

(a) For any given $0 < \rho < 1$, show that $P_n(z)$ has no zeros in $D(0, \rho)$ for n sufficiently large.

(b) Show that all of the zeros of $P_n(z)$ lie on $\partial D(0, 1)$.