Complex Variables Spring 2008
MS/PhD Qualifying Examination

Instruction: Complete all problems.

1) Let $f(z) = \log(z)$ denote the main branch of the complex logarithm, which is defined and holomorphic in

$$\mathbb{C} \setminus (-\infty, 0] \ .$$

In which domain is the function

$$g(z) = \log(\log(z))$$

holomorphic?

2) Let

$$U = \{z \in \mathbb{C} \ : \ 0 < |z| < 1\}$$

denote the open unit disk with the origin removed and let f denote a holomorphic function on U which has a pole of order three at the origin. Prove or disprove the following statements:

a) There is a constant $C > 0$ with

$$|f(z)| \le \frac{C}{|z|^3} \quad \text{for} \quad 0 < |z| \le \frac{1}{2} \ .$$

b) There is a constant $c > 0$ with

$$|f(z)| \ge \frac{c}{|z|^3} \quad \text{for} \quad 0 < |z| \le \frac{1}{2} \ .$$

c) There is a constant $C > 0$ with

$$|f(z)| \le \frac{C}{|z|^3} \quad \text{for} \quad 0 < |z| < 1 \ .$$

3) Let f and g be holomorphic functions defined for all z with $|z - c| < r$. Assume that

$$g(c) = g'(c) = 0, \quad g''(c) \ne 0 \ .$$

Show that the residue of the function

$$h(z) = \frac{f(z)}{g(z)}, \quad 0 < |z - c| < \varepsilon \ ,$$

is given by

$$\frac{n_1 f'(c) g''(c) + n_2 f(c) g'''(c)}{n_3 (g''(c))^2}$$

where n_1, n_2, n_3 are integers. Determine the integers n_j.

4) Evaluate

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2 - 2x + 2}\, dx \ .$$

5) Show that for any positive integer n all roots of

$$(1+z)^n + z^n = 0$$

lie on the line $x = -\frac{1}{2}$.

6) Let

$$f(0) = 0 \quad \text{and} \quad f(x+iy) = u(x,y) + iv(x,y) \quad \text{for} \quad z = x + iy \neq 0$$

where

$$u(x,y) = \frac{x^3 - y^3}{x^2 + y^2}, \quad v(x,y) = \frac{x^3 + y^3}{x^2 + y^2} \ .$$

a) Are the Cauchy–Riemann equations satisfied at $z = 0$?
b) Does the complex derivative $f'(0)$ exist?

7) Evaluate

$$\int_0^{2\pi} \frac{dt}{5 + 4\sin t} \ .$$

8) Determine the number of roots of the equation

$$z^3 - z^2 + 3z + 5 = 0$$

in the open right half–plane ($\mathrm{Re}\, z > 0$).