Complex Variables Fall 2008
MS/PhD Qualifying Examination

Instruction: Complete all problems.

1a) Calculate $w = (\sqrt{3} + i)^6$.
1b) Calculate the principle value of $w = (1 + i)^{4i}$ and write it in the form

$$w = r(\cos\alpha + i\sin\alpha), \quad r > 0, \quad \alpha \in \mathbb{R} \ .$$

2) Let

$$f(z) = \frac{e^z - 1}{z^4}, \quad z \neq 0 \ .$$

a) Evaluate

$$\int_\Gamma f(z)\, dz$$

when Γ is the positively oriented circle $|z| = 2$.
b) Determine the Laurent expansion of $f(z)$ valid for $z \neq 0$.

3) Find the Laurent expansion of

$$f(z) = \frac{5z}{(z+2)(z-3)}$$

which is valid for

$$2 < |z| < 3 \ .$$

4) Let $f(z)$ denote a function which is holomorphic in an open connected set U and for which $|f(z)|$ is constant in U. Can one conclude that $f(z)$ is constant in U? Justify your answer.

5) State Rouché's theorem and use it to prove that the equation

$$az^n = e^z$$

has n zeros (counted according to multiplicity) in $|z| < 1$ if

$$a \in \mathbb{C}, \quad |a| > e, \quad n \in \{1, 2, \ldots\} \ .$$

6) Let Γ denote the positively oriented unit circle. Evaluate

$$\int_\Gamma \frac{e^z}{z}\, dz$$

and use the result to evaluate

$$\int_0^{2\pi} e^{\cos t} \cos(\sin t)\, dt \quad \text{and} \quad \int_0^{2\pi} e^{\cos t} \sin(\sin t)\, dt \ .$$

7) Let $f(z)$ denote an entire function and assume that $f''(z)$ is bounded. Can you prove that $f(z)$ has the form

$$f(z) = az^2 + bz + c \ ?$$

8) Recall that the Γ–function is defined by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt$$

for suitable complex numbers z. a) In which z–region does the above formula define $\Gamma(z)$ as an analytic function? Justify your answer.

b) In which region of the complex plane can one continue Γ as an analytic function? Justify your answer and construct an analytic continuation of the function defined by the integral.