Complex Analysis Qualifying Examination

January 2009

Instructions: Please do the **eight** problems listed below. You may choose to answer the problems in any order. However, to help us in grading your exam please make sure to:
i. Start each question on a new sheet of paper.
ii. Write **only on one side** of each sheet of paper.
iii. Number each page and write the last four digits of your **UNM ID #** on each page.

1. (a) Show that $f(z) = -\dfrac{i}{2\cos z}$ and $g(z) = \dfrac{\sin z}{1+e^{i2z}}$ have the same poles and principal parts.
   (b) Find an entire function $\varphi(z)$ such that $\varphi(z) = -\left[\dfrac{i}{2\cos z} + \dfrac{\sin z}{1+e^{i2z}}\right]$.

2. Find the Laurent expansion for $f(z) = \dfrac{z^3+2z-1}{z^2-1}$ valid in the domain D, $D = \{z \in \mathbb{C} : |z| > 1\}$.

3. (a) If f is analytic on $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$, show that f is constant.
   (b) If f is analytic on $\mathbb{C}$ and satisfies $\max\{|f(z)| : |z| = r\} \le Mr^n$ for a fixed $M > 0$, $n > 0$, and a sequence of values $r = r_k$, with $r_k \to \infty$ as $k \to \infty$, show that f is a polynomial of degree less or equal to n.

4. (a) State Rouche's theorem.
   (b) If $f(z) = 4z^4 + 13z^2 + 3$, find the number of zeros of $f(z)$ inside the circle $\{z : |z| = 1\}$ and the number of zeros inside the annulus $\{z : 1 < |z| < 2\}$.

5. Use the residue theorem to evaluate
$$\int_0^{2\pi} \frac{d\theta}{4 + 3\cos\theta}.$$

6. Prove in complete detail: If f is analytic on an open set containing $\{z : \text{Im } z \le 0\}$ except for a finite number of singularities (none on the real axis), and $\lim_{z\to\infty} f(z) = 0$ (with $\text{Im } z \le 0$), and if $m < 0$, then
$$\lim_{R\to\infty} \int_{-R}^{R} f(x)e^{imx}\, dx = -2\pi i \sum_k \text{res } \left(f(z)e^{imz}, z = z_k\right)$$
   where z_k are the singularities of f in the lower half-plane.

7. (a) Where is the function $w = \cos z$ conformal?
   (b) Find the image of the domain $D = \{z = x + iy : -\pi/2 < x < \pi/2, 0 < y < \infty\}$ under the map $w = \cos z$.

8. Assume that $\{f_n(z)\}_{n=1}^{\infty}$ is a sequence of entire functions that converges to $f(z)$ uniformly on compact sets. Prove that $f(z)$ is entire and that for any $z_0 \in \mathbb{C}$ and $k \in \mathbb{N}$, $f_n^{(k)}(z_0) \to f^{(k)}(z_0)$ as $n \to \infty$