COMPLEX ANALYSIS QUALIFYING EXAM JANUARY 2010
DEPARTMENT OF MATHEMATICS AND STATISTICS
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Instructions: Complete all problems to get full credit. Start each problem on a new page, number the pages, and put only your banner identification number on each page. Clear and concise answers with good justification will improve your score.

1) Use residue theory to evaluate the integral
$$\int_0^{2\pi} \frac{d\theta}{4 + 3\cos\theta}.$$

2) If $f(z)$ is an entire function which is real valued on the real axis and has purely imaginary values on the imaginary axis, show that $f(z)$ is an odd function.

3) Expand
$$f(z) = \frac{2z^2}{z-2}$$
in powers of $(z-2)$. What is the radius of convergence of this series?

4) (a) State Rouche's theorem.
(b) Let $f(z)$ be a holomorphic function on an open set containing the unit disk $\mathbb{D} = \{z : |z| \leq 1\}$. If $f(z) < 1$ when $|z| = 1$ show that f has exactly one fixed point inside the unit circle, i.e., there is exactly one point z_o inside the unit circle such that $f(z_o) = z_o$.

5) Let $f(z)$ be an entire function, which is univalent on the complex plane, i.e., $f(z_1) = f(z_2)$ implies $z_1 = z_2$. Show that $f(z) = az + b$ for some a, $b \in \mathbb{C}$.

6) Let f be an entire function.
(a) If the real part of f is bounded from above, i.e., $\sup_{z\in\mathbb{C}} \Re f(z) \leq M < \infty$ show that f is a constant function.
(b) If $\Re f(z) \leq \Im f(z)$ for all $z \in \mathbb{C}$ show that f is a constant function.

7) (a) Show that if u is a harmonic function in a simply connected domain Ω of the complex plane then
$$u(z) = \ln|f(z)|$$
for some nowhere vanishing holomorphic function f on Ω.
(b) Find the harmonic conjugate of
$$u(x,y) = \frac{1}{2}\ln(x^2 + y^2)$$
on the domain $\mathbb{C} \setminus [0, +\infty)$.

8) The function f is holomorphic on $\mathbb{C} \cup \infty$ except for poles of order one at $z = -1$ and a pole of order two at $z = 1$, where the residues are 1 and -1 respectively. Determine $f(z)$ if $f(0) = 7$ and $f(2) = 13/3$.