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COMPLEX ANALYSIS QUALIFYING EXAM
TU. 9-12, JAN. 10, 2012

Department of Mathematics and Statistics
University of New Mexico

Instructions: Complete all problems to get full credit. Start each problem on a new page, number the pages, and put only your banner identification number on each page. Use only one side of each sheet.

Clear and concise answers with good justification will improve your score.

1) Let

$$p(z) = a_0 + a_1 z + \ldots + a_n z^n , \quad a_n \neq 0 ,$$

denote a polynomial of degree n and let Γ_r denote the positively oriented circle of radius r, centered at the origin.

Determine

$$\lim_{r\to\infty} \int_{\Gamma_r} \frac{p'(z)}{p(z)} \, dz .$$

2) Let Γ denote the circle of radius 2 centered at i. Describe the image of Γ under the map $z \to \frac{1}{z}$. What kind of curve is it?

3) Let

$$f(z) = \sum_{n=1}^{\infty} \frac{z^n}{1-z^n} \quad \text{for} \quad |z| < 1 .$$

Evaluate $f^{(10)}(0)$. You may leave factorials in your answer without evaluating them.

4) Find the first four terms of the Laurent series representation of the function

$$f(z) = \frac{e^z}{z(z^2+1)}$$

for $0 < |z| < 1$.

5) Let Γ_2 denote the positively oriented circle of radius 2, centered at the origin. Evaluate

$$\int_{\Gamma_2} \tan z \, dz .$$

6) Classify the singularities and find the associated residues for the following functions

a) $f(z) = \frac{\sin^2(2z)}{z^4}$

b) $g(z) = \frac{z^2+2z-1}{(z-1)(z+2)^2}$

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7) Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{x \sin x}{(x^2+1)(x^2+4)}\, dx$$

and justify your computation.

8) The Gamma function can be defined by

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t}\, dt$$

for suitable complex numbers z.

a) In which region R of the complex plane does the integral define an analytic function? Why?

b) Derive the functional equation for the Gamma function:

$$\Gamma(z+1) = z\Gamma(z), \quad z \in R$$

c) Use the functional equation to obtain an analytic continuation of the Gamma function.

COMPLEX ANALYSIS QUALIFYING EXAM F 2012

We, Aug. 15, 9–12
Department of Mathematics and Statistics
University of New Mexico

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Clear and concise answers with good justification will improve your score.

1) Let Γ denote the positively oriented circle of radius 3 centered at the origin and let

$$f(w) = \int_\Gamma \frac{z^2 + z - 2}{z - w}\, dz, \quad |w| \neq 3 .$$

Evaluate
a) $f(2)$;
b) $f(-2)$;
c) $f(4)$.

2) Find all solutions $z = x + iy$ of the equation

$$\sin z = 2$$

and sketch the solutions z as points in the complex plane.

3) a) Sketch the region in the complex plane consisting of all z with

$$|z - 4| > |z| .$$

b) Sketch the region in the complex plane consisting of all z with

$$Im(z^2) > 0 .$$

4) Evaluate

$$\int_0^\infty \frac{\cos x}{x^2 + 1}\, dx$$

and justify your computation.

5) Let Γ denote the straight line from the origin to the point $\pi + 2i$. Evaluate

$$\int_\Gamma \cos(z/2)\, dz .$$

6) a) Find the bilinear transformation

$$w = f(z) = \frac{az + b}{cz + d}$$

for which

$$f(0) = 0, \quad f(2) = 4, \quad f(i) = 1 - i .$$

b) Find the fixed points of the transformation.
c) What is the image of the region $\{y \geq 1\}$ under the transformation?

7) a) Show: If t is real and

$$z = \frac{1+it}{1-it}$$

then $|z| = 1$.

b) Let z be any complex number with $|z| = 1, z \neq -1$. Show that one can write

$$z = \frac{1+it}{1-it}$$

for some real t. Is t unique?

8) Let w denote an n–th root of unity, $w \neq 1$. Evaluate

$$1 + 2w + 3w^2 + \ldots + nw^{n-1} \ .$$

(Find a simple expression for the sum which is based on the geometric sum formula.)

COMPLEX ANALYSIS QUALIFYING EXAM JANUARY 2011
DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF NEW MEXICO

Instructions: Complete all problems to get full credit. Start each problem on a new page, number the pages, and put only your banner identification number on each page. Clear and concise answers with good justification will improve your score.

1) Find the Laurent series of the function

$$f(z) = \frac{2z^2}{(z-1)(z-3)}$$

on the indicated domain D.
a) D is the annulus $D = \{z : 1 < |z| < 3\}$.
b) D is the exterior of the disc $D(0,3)$, i.e., $D = \{z : |z| > 3\}$

2) Let Γ denote the positively oriented unit circle. Evaluate

$$\int_\Gamma \frac{e^z - 1}{\sin^3 z}\, dz\ .$$

3) Let $f(z)$ denote a function which is holomorphic in $\mathbb{C} \setminus \{0\}$ and has the Laurent expansion $f(z) = \sum_{j=-\infty}^{\infty} a_j z^j$. Assuimg that $f(z)$ is real for all real z, does it follow that all coefficients a_j are real? Give a proof or counterexample.

4) Determine the number of zeros in the half-plane $\Re z < 0$ of the function

$$f(z) = e^z - z - 2.$$

5) Determine the region on which $f(z) = \prod_{k=1}^{\infty}\left(1 + z^k\right)$ defines a holomorphic function.

6) Let u be a harmonic function on the unit disc $D = \{z : |z| < 1\}$, which is the real part of the holomorphic function f on D. Show that for any $0 < r < 1$ we have

$$f(z) = \frac{1}{\pi i}\oint_{|\zeta|=r} \frac{u(\zeta)}{\zeta - z}\, d\zeta - \overline{f(0)} \qquad \text{for } |z| < r.$$

7) Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions holomorphic on the open unit disc $D = \{z : |z| < 1\}$ and continuous on the closed unit disc $\bar{D} = \{z : |z| \le 1\}$. Show that if the sequence $\{f_n\}_{n=1}^{\infty}$ converges uniformly on the unit circle $\partial D = \{z : |z| = 1\}$, then there is a function f such that
(i) $f_n \to f$ on D, i.e., $\lim_{n\to\infty} f_n(z) = f(z)$ for every $z \in D$, and
(ii) f is holomorphic on D.

8) Let f and g be entire functions such that

$$|f(z)| \le |g(z)|, \qquad z \in \mathbb{C}.$$

Show that there is a constant λ such that $f = \lambda g$.

COMPLEX ANALYSIS QUALIFYING EXAM AUGUST 2011
DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF NEW MEXICO

Instructions: Complete all problems to receive full credit. Start each problem on a new page, number the pages, and put only your banner identification number on each page. Clear and concise answers with good justification will improve your score.

Everywhere $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ denotes the open unit disc and $\partial\mathbb{D}$ is its boundary.

1) Use the calculus of residues to compute the following integrals $\frac{1}{2\pi i}\oint_\gamma f(z)\,dz$, where
$$f(z) = e^z / [z(z+2)]$$
and γ is the negatively oriented triangle with vertices $1 \pm i$ and -4.

2) Classify the singularities of the function $f(z) = \frac{\sin z}{z^3}$ on $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ and find the principal (singular) part at each of the singular points.

3) Suppose f is a doubly periodic meromorphic function on $\mathbb{C}$ with periods 1 and i so that f is holomorphic on $\mathbb{C} \setminus \{m + ni \mid m, n \in \mathbb{Z}\}$ (i.e. the poles of f are the points of the integer lattice). Prove that the residue of f at each of its poles is zero.

4) Find the number of roots of the polynomial $p(z) = z^7 + 6z^3 + 7$ in the first quadrant by showing, among other things, that $p(Re^{it}) = R^7 e^{7it}\left(1 + \zeta\left(Re^{it}\right)\right)$ where the function ζ satisfies
$$\lim_{R\to\infty} \zeta(Re^{it}) = 0.$$

5) Let $\mathbb{D}^\times = \mathbb{D} \setminus \{0\}$. Prove the following refined versions of Riemann's removable singularity theorem. If f is holomorphic on $\mathbb{D}^\times$ and $\lim z\, f(z) = 0$ when $z \to 0$ then 0 is a removable singularity.

6) a) Let u be a bounded continuous function on the set $\mathbb{C}\setminus\mathbb{D}$ which is harmonic on the complement of the closed unit disc $\mathbb{D}$. Show that if u vanishes on $\partial\mathbb{D}$ then for any given $\epsilon > 0$ we have
$$|u(z)| \le \epsilon \ln |z|, \qquad z \in \mathbb{C} \setminus \mathbb{D}.$$
Prove that $u(z) = 0$ for all $z \in \mathbb{C} \setminus \mathbb{D}$.

   b) Show that the Dirichlet problem for the Laplace operator on the complement of the unit disc $\mathbb{D}$ with boundary data on $\partial\mathbb{D}$ given by a continuous function f has a unique *bounded* solution.

7) a) Let Ω be an open subset of $\mathbb{C}$. Suppose that K is a compact set contained in the open set U whose closure $\bar{U}$ is a compact contained in Ω. Show that there exists a constant $C = C(K, U, \Omega)$ such that for any $f \in A(\Omega)$ we have
$$\max_{z\in K} |f'(z)| \le C \max_{z\in U} |f(z)|.$$
It might be usefulto consider the case of a disc K and a twice larger concentric disc U.

   b) Let $\mathfrak{F}$ be a normal family of holomorphic functions on U. Show that the family $\mathfrak{F}'$ of all complex derivatives of the functions in $\mathfrak{F}$ is also a normal family, i.e., every sequence of $\mathfrak{F}$ has a subsequence that converges uniformly on all compact subsets of U.

8) Show that a function f which is holomorphic in a neighborhood of the unit circle γ centered at the origin can be approximated uniformly by polynomials if and only if there exists a disc D containing the circle to which f can be extended as a holomorphic function, i.e., f is the restriction of a function holomorphic on D.

Complex Variables Fall 2010
MS/PhD Qualifying Examination

Instruction: Complete all problems. Justify your answers.

1) Let a and z denote complex numbers with $|a| < 1$ and $|z| < 1$. Does it follow that

$$\left|\frac{z-a}{1-\bar{a}z}\right| < 1 \ ?$$

2) Let $f(z) = u(z) + iv(z)$ denote an entire function where $u(z)$ and $v(z)$ are real valued. Assume that

$$u(z) < 5 \quad \text{if} \quad |z| > 5 \ .$$

Does it follow that $f(z)$ is constant?

3) Let Γ denote the circle with parametrization $z(t) = 1 + 2e^{it}, 0 \le t \le 2\pi$. Determine

$$\int_\Gamma \frac{e^z}{z^3}\, dz \ .$$

4) Determine the residue of the function

$$F(z) = \frac{e^{(z^2)}}{(z-i)^3}$$

at its pole.

5) State a version of Rouché's theorem and use it to determine the number of solutions of the equation

$$\sin(z) + \cos(z) = 6z \quad \text{with} \quad |z| < 1 \ .$$

6) Let Γ denote the positively oriented unit circle. Evaluate

$$\int_\Gamma \frac{z}{\sin(z^2)}\, dz \ .$$

7) Consider the function

$$f(z) = \frac{1}{\sin\left(\frac{1}{1-z}\right)} \ .$$

Does $f(z)$ have a pole at $z = 1$? Does $f(z)$ have an essential singularity at $z = 1$?

8) Define the function

$$f(z) = \frac{\sin(\sqrt{z})}{\sqrt{z}}$$

by

$$f(z) = \frac{\sin(\sqrt{z})}{\sqrt{z}} = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)!} z^j \ .$$

a) Is $f(z)$ entire?

b) Define the order of growth of $f(z)$ and determine it.

c) If $p(z)$ and $q(z)$ are non–trivial polynomials, what can you say about the number of solutions of the equation

$$p(z) f(z) = q(z) \ ?$$