Complex Analysis Qualifying Exam

August 2013

Instructions: Solve any 8 problems to get full credit. Start each problem on a new page, number the pages, and put the last four digits of your banner id on each page.

Justify all your steps and explicitly verify the assumptions of every theorem you apply. Clear and concise answers will improve your score.

1. Let $f(z) = \dfrac{z^3 e^{1/z}}{z^2+1}$.

   (a) Find and classify all the singularities of $f(z)$ in the complex plane and at infinity.

   (b) Does $|f(z)|$ diverge to infinity as $|z|$ goes to zero? Justify your answer.

   (c) Find the first three terms of the Laurent series of f in the domain $\{z \in \mathbb{C} : |z| > 1\}$.

2. Evaluate $\displaystyle\oint_\Gamma \frac{z^3 e^{1/z}}{z^2+1} dz$, where $\Gamma = \{z \in \mathbb{C} : |\mathrm{Re}\, z| + |\mathrm{Im}\, z| = 3\}$. Fully simplify your answer.

3. Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{2x - \pi} dx$. Provide a detailed solution.

4. Find all entire functions $f(z)$ such that $|f(z)| \geq |\sin z|^{10}$ for all $z \in \mathbb{C}$.

5. Let a nonconstant function $f(z)$ be analytic on a bounded domain Ω and continuous on its closure $\overline{\Omega}$. If $|f(z)|$ is constant on the boundary $\partial\Omega$, prove that $f(z)$ has a zero in Ω.

6. Find a function $\phi(z)$ harmonic on $\Omega = \{z \in \mathbb{C} : |z| < 2,\ |z-1| > 1\}$ such that $\phi = 1$ on the outer boundary $\{z \in \mathbb{C} : |z| = 2,\ z \neq 2\}$ and $\phi = 3$ on the inner boundary $\{z \in \mathbb{C} : |z-1| = 1,\ z \neq 2\}$. [Hint: map Ω to a simpler domain by a Möbius transformation.]

7. Find how many solutions the equation $\sin z = e z^3$ has on the unit disc. Justify your answer. Provide detailed estimates in your solution.

8. (a) Is there an entire function $f(z)$ such that $f(0) = 1$ and $f(\frac{n^2-1}{n^2}) = 0$ for all integers $n \geq 2$? If yes, find it.

   (b) Is there an analytic function $g(z)$ on the open unit disk such that $g(0) = 1$ and $g(\frac{n^2-1}{n^2}) = 0$ for all integers $n \geq 2$? If yes, find it.

9. Show that the equation $\dfrac{\sin\sqrt{z}}{\sqrt{z}} = c$ has a solution for **every** $c \in \mathbb{C}$. [Hint: proceed by contradiction using growth order considerations.]

10. Prove that a continuous function $f(z)$ on the unit circle $\partial\mathbb{D}$ can be uniformly approximated on $\partial\mathbb{D}$ by polynomials if and only if $f(z)$ has an analytic extension to $\mathbb{D}$ (i.e., there exists $F(z)$ analytic on $\mathbb{D}$, continuous on $\overline{\mathbb{D}}$, and such that $F(z) = f(z)$ on $\partial\mathbb{D}$).