COMPLEX ANALYSIS QUALIFYING EXAM
MONDAY 9-12, JAN. 13, 2014

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Instructions: Complete all problems to get full credit. Start each problem on a new page, number the pages, and put only your banner identification number on each page. Use only one side of each sheet.

Clear and concise answers with good justification will improve your score.

1) Let Γ denote the positively oriented circle of radius 2 centered at the origin. Evaluate

$$\int_\Gamma \frac{dz}{z^n(1-z)} \quad \text{for} \quad n = 0, 1, 2, \ldots$$

2) Let

$$p(w) = \sum_{j=0}^{J} a_j w^j, \quad a_J \neq 0 \ ,$$

denote a polynomial of degree $J \geq 1$. Determine the radius of convergence of the power series

$$\sum_{n=0}^{\infty} p(n) z^n \ .$$

3) For the equation $e^z - z = 2014$ in the left half-plane $\{z \in \mathbb{C} : \mathrm{Re}\,(z) < 0\}$ show that: (a) there exists exactly one solution, (b) the solution is real.

Hint: Use Rouché's Theorem.

4) Let

$$f(z) = \frac{1}{\sin(1/z)}$$

be defined for all complex $z \neq 0$ with $\sin(1/z) \neq 0$.

Can you determine the residue of f at $z = 0$? Does this make sense? Justify your answer.

5) Determine if the function $f(z) = i|z|^2 - 2z\mathrm{Im}\,(z)$ is entire. If it is, find the Taylor series expansion of $f(z)$ centered at $z_0 = 1$.

6) Let $f(z) = \dfrac{e^{1/z}}{1-z}$. (a) Find and classify all the singularities of $f(z)$ on $\mathbb{C}$. (b) Does $|f(z)|$ diverge to infinity as $|z|$ goes to zero? (c) Show that

$$\int_\Gamma f(z)dz = 2\pi i(e-1)$$

where Γ is the positively oriented circle of radius $1/2$ centered at the origin.

7) Suppose f is holomorphic on $\Omega = \{z \in \mathbb{C} : 0 < |z| < 1\}$ and $\operatorname{Re}(f) \geq -2014$ on Ω. Prove that f has a removable singularity at the origin.

8) Let Γ denote the circle of radius 2 centered at $z_0 = 1$. Determine the image of Γ under the transformation

$$z \to \frac{1}{z} \, .$$