COMPLEX ANALYSIS QUALIFYING EXAM
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Instructions: Complete all problems to get full credit. Start each problem on a new page, number the pages, and put only your banner identification number on each page. Use only one side of each sheet.

Clear and concise answers with good justification will improve your score.

1) Let

$$u(x,y) = -2xy + e^x \sin y \quad \text{for} \quad (x,y) \in \mathbb{R}^2 \ .$$

a) Prove that $u(x,y)$ is a harmonic function and determine a harmonic conjugate $v(x,y)$.
b) Determine a simple expression for an entire function $f(z)$ with

$$u(x,y) = \operatorname{Re} f(x+iy), \quad v(x,y) = \operatorname{Im} f(x+iy) \ .$$

2) Let H denote the half–plane

$$H = \{z = x + iy \ : \ x > \frac{1}{2}, y \in \mathbb{R}\} \ .$$

Determine the image of H under the mapping $z \to \frac{1}{z}$ and sketch the image.

3) Compute the integral

$$\int_\Gamma \frac{dz}{z^3(z+4)}$$

if
a) Γ is the positively oriented circle $|z| = 2$.
b) Γ is the positively oriented circle $|z+2| = 3$.

4) Consider the series

$$f(z) = \sum_{j=0}^{\infty} (j+1)(z+1)^j \ .$$

a) Determine a circle, as large as possible, where the series defines a holomorphic function.
b) Evaluate the series and show that $f(z)$ has a holomorphic extension in $\mathbb{C} \setminus \{0\}$.

5) Consider the function $f(z) = z\cos\left(\frac{\pi}{z}\right)$. Does $f(z)$ have a finite or infinite limit as z goes to zero? Justify. Evaluate the residue of $f(z)$ at $z = 0$.

6) Determine how many solutions of $z^7 - 7z^4 + z^3 - 3z = 1$ are in the annulus $1 < |z| < 2$.

7) Find all entire functions $f(z)$ that satisfy $|f(z)| \le e^{\operatorname{Im}(z)}$ for all $z \in \mathbb{C}$.

8) Suppose $f(z)$ is a function such that $f(z)$ is analytic on $|z| < 1$ with $|f(z)| < 1$ for all $|z| < 1$. Let $f(0) = \alpha \neq 0$. Prove that $f(z) \neq 0$ for $|z| < |\alpha|$.