COMPLEX ANALYSIS QUALIFYING EXAM
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Instructions: Complete all problems to get full credit.

Start each problem on a new page, number the pages, and put only your banner identification number on each page.

Use only one side of each sheet.

Clear and concise answers with good justification will improve your score.

1) (a) Define the function $\log w$ in a suitable subdomain of the complex plane and then prove that the function

$$\log\left(\frac{1}{1-z}\right), \quad |z| < 1 ,$$

is holomorphic in the open unit disc.

(b) Compute the coefficients a_j in the expansion

$$\log\left(\frac{1}{1-z}\right) = \sum_{j=0}^{\infty} a_j z^j, \quad |z| < 1 .$$

(c) Does the function $\log(1/(1-z))$ have a Laurent expansion for $|z| > 1$? Justify your answer.

2) Let Γ denote the positively oriented circle of radius 1 centered at the origin. Compute the three integrals

$$\int_\Gamma \frac{dz}{(\sin z)^j} \quad \text{for} \quad j = 1, 2, 3 .$$

3) Let $f(z)$ denote a function which is holomorphic in a neighborhood of the origin. Is it possible that

$$|f^{(n)}(0)| \geq (n!)^2$$

for all large integers n? Justify your answer.

4) Use the residue theorem to evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^2}$.

5) (a) State the open mapping theorem.

(b) Prove that if $f(z) = u(z) + iv(z)$ is analytic on the unit disc $D = \{z \in \mathbb{C} \ : \ |z| < 1\}$ (with real valued functions $u(z)$ and $v(z)$) and $u(z) + v(z) = 1$ for all $z \in D$, then $f(z)$ is constant in D.

6) (a) Find the Möbius transformation

$$w = \frac{az+b}{cz+d}$$

that sends $z = -1$ to $w = -1$, $z = 1$ to $w = 1$, and $z = i$ to $w = -i$.

(b) Find the image of the half disc $G = \{z \in \mathbb{C} \ : \ |z| < 1, \mathrm{Re} z > 0\}$ under the Möbius transformation found in part (a). Plot G and $f(G)$.

7) a) Where is the transformation $w = f(z) = z + \frac{1}{z}$ conformal?
b) Find the image of the first quadrant $G = \{z \in \mathbb{C} \ : \ \mathrm{Re} z > 0, \mathrm{Im} z > 0\}$ under the mapping $w = f(z)$. Plot G and $f(G)$.

8) (a) State the Mittag-Leffler theorem.
(b) Find a meromorphic function $f(z)$ with simple poles at $z = n$, $n = 1, 2, 3, \ldots$, and with $\mathrm{res} f(z)|_{z=n} = n$.