

# COMPLEX ANALYSIS QUALIFYING EXAM

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*Instructions:* Complete all problems to get full credit. Start each problem on a new page, number the pages, and put only your code word (not your banner ID number) on each page. Use only one side of each sheet.

Clear and concise answers with good justification will improve your score.

- 1) Show that  $|\operatorname{Im}(z^k)| \leq k |\operatorname{Im}(z)| |z|^{k-1}$  for all complex numbers  $z$  and positive integers  $k$ .  
Hint: You may first try to prove the inequality in the special case  $|z| = 1$ .

- 2) Find a Möbius transformation that simultaneously maps the unit circle and the line  $x + y = 0$  into the coordinate axes  $x = 0$  and  $y = 0$ , respectively.

- 3) Let  $p(z)$  be a polynomial of degree  $k \geq 1$ . Determine the radius of convergence of the power series  $\sum_{n=0}^{\infty} p(2^n) z^{2n}$ .

- 4) Let  $f(z)$  be an analytic function on  $0 < |z| < 2$  such that  $f((-1)^n/n) = (-1)^n$  for all positive integers  $n$ . Show that  $\inf_{0 < |z| < 2} |f(z)| = 0$ .

- 5) Show that  $\int_0^{\infty} \frac{dx}{\sqrt{x}(x^2 + 1)} = \frac{\pi}{\sqrt{2}}$ . Provide a detailed justification.

- 6) Let  $\Gamma$  be the positively oriented unit circle. Evaluate  $\int_{\Gamma} \frac{dz}{z^9 - 3z^4 + 5z}$ .

Hint: First show that the function has only one singularity in the unit disc.

- 7) Let  $f(z)$  be an entire function such that  $|f(z)| \leq 1$  on the unit circle and  $f(\alpha) = 0$  for some  $\alpha$  in the unit disc  $\mathbb{D}$ . Prove that  $|f(z)| \leq \left| \frac{z - \alpha}{1 - \bar{\alpha}z} \right|$  for all  $z \in \mathbb{D}$ .

- 8) Construct a function  $f(z)$  which is analytic on  $\mathbb{C}$  except at positive integers, has a zero at  $-1$ , has a simple pole at every positive integer  $n$ , and has a residue 1 at every pole. Justify that your function has the required properties. Are there any other functions satisfying these properties? If so, how many?