COMPLEX ANALYSIS QUALIFYING EXAM

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Instructions: Complete all problems to get full credit. Start each problem on a new page, number the pages, and put only your code word (not your banner ID number) on each page. Use only one side of each sheet.

Clear and concise answers with good justification will improve your score.

1. a) Show that the Calculus version of Mean-Value Theorem does not hold for $f(z) = e^z$, that is, there exist $z, w \in \mathbb{C}$ such that $f(z) - f(w) \neq f'(c)(z - w)$ for all $c \in \mathbb{C}$.

   b) Prove that if $\operatorname{Re}(z) \leq 0$ and $\operatorname{Re}(w) \leq 0$, then $|e^z - e^w| \leq |z - w|$.

2. Find a Möbius transformation that maps the region $\{z : \ |z-1| > 1, \ |z-2| < 2\}$ to $\{w : \ |\operatorname{Re}(w)| < 1\}$.

3. Let f be a nonconstant entire function such that $f(1+z) = 1 + f(z)$ for all $z \in \mathbb{C}$. Determine the range of f, that is, the set $f(\mathbb{C})$.

4. Let $f(z)$ be an entire function such that $|f(z)| \geq \sqrt{|z|}$ for all $|z| \leq 1$. Prove that $|f(0)| \geq 1$.

5. Suppose f is a function analytic on $\mathbb{D}$ and such that $\displaystyle\sum_{k=0}^{\infty} \frac{f^{(k)}(i/2)}{(k+1)^{2017}}$ converges. Prove that f is an entire function.

6. Let f be an entire function such that $|f(z)| \leq |\operatorname{Im}(z)|^3$ for all $z \in \mathbb{C}$. Show that f is identically zero.

7. Show that $\displaystyle\oint_{|z|=1/2} \frac{\cos(1/z)}{2 - \pi z}\, dz = -2i.$

8. Show that the equation $e^z = z$ has infinitely many solutions in $\mathbb{C}$.