COMPLEX ANALYSIS QUALIFYING EXAM
AUGUST 8 , 2018

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Instructions: Complete all problems to get full credit.

Start each problem on a new page, number the pages, and put your code identification number on each page.

Use only one side of each sheet.

Clear and concise answers with good justification will improve your score.

1) (a) State Schwarz Lemma.

   (b) Let $D = \{z \in \mathbb{C} \ : \ |z| < 1\}$ denote the open unit disk and let f denote a holomorphic function which maps D bijectively onto itself and satisfies

$$f(0) = 0 \quad \text{and} \quad f'(0) \text{ is real and positive .}$$

   Prove that

$$f(z) = z \quad \text{for all} \quad z \in D \ .$$

2) (a) Let f denote an entire function. Define the related function

$$g(z) = \bar{f}(\bar{z}) \quad \text{for} \quad z \in \mathbb{C} \ .$$

   Does it follow that g is also an entire function? Justify your answer.

   (b) Let f denote an entire function which is real on the real line. Does it follow that

$$f(\bar{z}) = \bar{f}(z) \quad \text{for all} \quad z \in \mathbb{C} \ ?$$

   Justify your answer.

3) Let f be a function analytic in the open unit disc D and continuous in the closure of D. Prove that

$$\int_0^1 f(x)dx = \frac{1}{2\pi i} \oint_{|z|=1} f(z) \log(z) dz$$

   where log denotes the branch of the logarithm whose imaginary part takes values in $[0, 2\pi)$.

4) Find a conformal map from the disk D with a slit $\{|z| < 1\} \setminus [0, 1]$ to the strip $\{|\text{Re } z| < 1\}$.

5) Let $f(z)$ be an entire function which is real on the real axis and purely imaginary on the imaginary axis. Prove that $f(z)$ is an odd function, i.e.,

$$f(-z) = -f(z) \quad \text{for all} \quad z \in \mathbb{C} \ .$$

6) Prove that $(3 - e)\,|z| < |1 - e^z| < (e - 1)\,|z|$ when $0 < |z| < 1$.

7) Suppose f is an analytic function on the punctured unit disk $D \setminus \{0\}$ such that $f(\frac{1}{n}) = \frac{(-1)^n}{n}$ for all positive integers n. Prove that $\lim_{z \to 0} |f(z)|$ does not exist.

8) Show that the polynomial $z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_n$ has all its roots in the disk $|z| < 1 + \max\{|a_1|, |a_2|, \ldots, |a_n|\}$.