COMPLEX ANALYSIS QUALIFYING EXAM
JANUARY 9, 2019

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Instructions: At the top of each page you use, write the secret code that you shared with Ana and label all problems appropriately. Complete all problems to get full credit. Start each problem on a new page, number the pages. Use only one side of each sheet. Clear and concise answers with good justification will improve your score.

1) Let $f(z)$ denote a function which is holomorphic in
$$\Omega = \{z \in \mathbb{C} \ : \ 0 < |z| < 1\} \ .$$
Prove or disprove the following: There exists a function $F(z)$, holomorphic in Ω, with $F'(z) = f(z)$ in Ω if and only if the residue of $f(z)$ at $z = 0$ equals zero.

2) Does the function $f(z) = z^4 + 6z + 3$ have a zero z_0 with $|z_0| \geq 2$? Justify your answer.

3) Let $f(z)$ and $g(z)$ denote entire functions satisfying
$$|f(z)| \geq 10|g(z)| \quad \text{for all} \quad z \in \mathbb{C} \ .$$
Does it follow that there exists $\lambda \in \mathbb{C}$ with
$$f(z) = \lambda g(z) \quad \text{for all} \quad z \in \mathbb{C} \ ?$$
Justify your answer.

4) For any complex numbers $z_1, \ldots, z_5$ in the quarter-plane $\{z \in \mathbb{C} : |\arg z| < \frac{\pi}{4}\}$ show that
$$\frac{1}{2}(|z_1|^2 + \cdots + |z_5|^2) \leq |z_1 + \cdots + z_5|^2 \leq 5(|z_1|^2 + \cdots + |z_5|^2) \ .$$

5) Evaluate
$$\int_{-\infty}^{\infty} \frac{\sin x}{x(x-\pi)} \, dx \ .$$
Justify the steps in your calculation.

6) Let $p(w)$ be a polynomial of degree d,
$$p(w) = \sum_{j=0}^{d} p_j w^j \ .$$
What can you say about the radius of convergence of the power series
$$\sum_{n=0}^{\infty} p(2^n) z^{2n} \ ?$$

7) Find a conformal map from the open unit disk with a slit
$$\{z \in \mathbb{C} : |z| < 1\} \setminus [-1, 0]$$
to the quarter-plane
$$\{z \in \mathbb{C} : \text{Re } z > 0, \ \text{Im } z > 0\} \ .$$

8) Let $f(z)$ be an analytic function on the punctured unit disk $\Omega = \{z \in \mathbb{C} : 0 < |z| < 1\}$ such that $|f(z)| \leq \log(1/|z|)$ for all $z \in \Omega$. Prove that $f(z)$ must be identically zero on Ω.