COMPLEX ANALYSIS QUALIFYING EXAM
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Instructions: At the top of each page you use, write the secret code that you shared with Ana and label all problems appropriately. Complete all problems to get full credit. Start each problem on a new page, number the pages. Use only one side of each sheet. Clear and concise answers with good justification will improve your score.

1) Let $f(z)$ denote an entire function satisfying the bound
$$|f(z)| \le C_1 + C_2|z|^{6+\alpha} \quad \text{for all} \quad z \in \mathbb{C}$$
where $C_1 > 0, C_2 > 0, 0 < \alpha < 1$ are constants independent of z. Prove that $f(z)$ is a polynomial of degree ≤ 6.

2) Consider a power series
$$\sum_{j=0}^{\infty} a_j z^j$$
and assume that there are constants $M > 0$ and $m > 0$ so that
$$|a_j| \le M \quad \text{for all} \quad j = 0, 1, 2, \ldots$$
and
$$|a_j| \ge m > 0 \quad \text{for infinitely many } j \ .$$
What can you say about the radius of convergence of the power series?

3) Evaluate
$$\int_{-\infty}^{\infty} \frac{\cos x}{1+x^2}\, dx \ .$$

4) Let $a \in \mathbb{C}, |a| > e$, and let n denote a positive integer. Prove that the equation
$$e^z = az^n$$
has n solutions z_j (not necessarily distinct) with $|z_j| < 1$.

5) Let Γ denote the circle of radius 2 centered at the point $z_0 = -1$. Determine the image of Γ under the map
$$f(z) = \frac{1}{z} \ .$$
Sketch Γ and its image.

6) Let $f(z)$ denote a function which is holomorphic in the open unit disc
$$U = \{z \in \mathbb{C} \ : \ |z| < 1\}$$
and satisfies the bound $|f(z)| \le 6$ for all $z \in U$. Derive an estimate for
$$\left| f^{(3)}\Big(\frac{1+i}{2}\Big)\right| \ .$$

7) Let $f(z) = \dfrac{z^3 e^{1/z}}{z^2+1}$.

(a) Find and classify all the singularities of $f(z)$ in the extended complex plane.

(b) Does $|f(z)|$ diverge to infinity as $|z|$ goes to zero? Justify your answer.
(c) Find the first two terms of the singular part of f in the domain $\{z \in \mathbb{C} : |z| < 1\}$.
(d) Find the first four terms of the Laurent series of f in the domain $\{z \in \mathbb{C} : |z| > 1\}$.
(e) Evaluate $\int_\Gamma f(z)dz$, where $\Gamma = \{z \in \mathbb{C} : |Rez| + |Imz| = 3\}$.

8) (a) Let $f(z)$ be holomorphic in the open set Ω. Show that

$$\Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\Big)|f(z)|^2 = 4|f'(z)|^2 \ .$$

Here, as usual, $z = x + iy$.

(b) Let $f_1(z), f_2(z), \ldots, f_n(z)$ denote holomorphic functions in the open connected set Ω. Show that

$$|f_1(z)|^2 + |f_2(z)^2 + \ldots + |f_n(z)|^2$$

is harmonic in Ω if and only if all the functions $f_k(z), k = 1, \ldots, n$, are constant in Ω.