COMPLEX ANALYSIS QUALIFYING EXAM
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Instructions: At the top of each page you use, write the secret code that you shared with Ana and label all problems. Complete all 8 problems to get full credit. Start each problem on a new page and number the pages. Use only one side of each sheet. Clear and concise answer with good justification will improve your score.

1. Evaluate the integral:
$$I = \int_0^2 \frac{\sqrt{x(2-x)}}{3-x}\mathrm{d}x.$$

2. Find a conformal map from the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ to the strip
$$\{w \in \mathbb{C} : 0 < \operatorname{Im} w < \pi\}.$$

3. Let n be a positive integer. Find all entire functions $f(z)$ for which there exists positive constants M, R such that
$$|f(z)| \geq M|z|^n, \qquad \text{whenever } |z| > R.$$

4. Suppose f is analytic on the disk $D(0, R)$ about the origin with radius $R > 1$ and that $|f(z)| < 1$ for $|z| = 1$. Prove that there exists a unique point z_0 in the unit disk such that $f(z_0) = z_0$.

5. Suppose f is an analytic function on an open set $U \subset \mathbb{C}$ with a zero of order k at a point $z_0 \in U$. Prove that there is a nonempty open disk $D(z_0, r)$ centered at z_0 and an analytic function g on $D(z_0, r)$ such that $f = g^k$ on the disk and $g'(z_0) \neq 0$.

6. Suppose f is a holomorphic function on $\mathbb{C} \setminus \{0\}$ which is even in that $f(-z) = f(z)$. Prove that f admits a holomorphic antiderivative on its domain, that is, there exists F holomorphic on $\mathbb{C} \setminus \{0\}$ such that $F'(z) = f(z)$ for all $z \neq 0$.

   Hint: What can you say about the residue of f at 0?

7. Suppose u is a real-valued, nonconstant, harmonic function on the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Prove that if $z_0 \in \mathbb{D}$, then there exists $\epsilon > 0$ such that
$$(u(z_0) - \epsilon, u(z_0) + \epsilon) \subset u(\mathbb{D}),$$
   that is, the image of u contains an open interval about $u(z_0)$. (This of course establishes that $u(\mathbb{D})$ is an open set.)

8. Prove that if f is a nonconstant entire function and $b^2 \neq 4ac$, then the function $g(z) := af^2(z) + bf(z) + c$ has a zero.