

COMPLEX ANALYSIS QUALIFYING EXAM

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Instructions: At the top of each page you use, write the secret code that you shared with Ana and label all problems appropriately. Complete all problems to get full credit. Start each problem on a new page, number the pages. Use only one side of each sheet. Clear and concise answers with good justification will improve your score.

1. Consider the function:

$$f(z) = \frac{z}{e^z - 1}$$

- a) Find first four terms of Taylor expansion around $z = 0$ and radius of convergence.
b) Find all coefficients expressed through recurrent relation (this can count as the first part of a solution of part a)).

2. Evaluate the definite integral:

$$I = \int_0^3 \frac{\sqrt{x(3-x)}}{4-x} dx.$$

You have to explain all the steps carefully (show contours, introduce branches etc.).

3. Expand the function

$$f(z) = \frac{1 + 2z^2}{z^2 + z^4}$$

into power series of z in all areas of convergence (you have to determine all of them).

4. Evaluate the improper integral:

$$I = \int_0^{+\infty} \frac{x^{1/3} \ln x}{x^2 + 4} dx.$$

You have to explain all the steps carefully (show contours, introduce branches etc.).

5. For the following domains Ω determine if Ω can be mapped analytically onto the unit disk $|z| < 1$. Justify your answer. If your answer is yes, find such a map.

a) $\Omega = \mathbb{C} \setminus \{0\}$.

b) $\Omega = \mathbb{C} \setminus [0, \infty)$.

6. Suppose f is an entire function such that $f(0) = 1$ and $\int_0^{2\pi} |f(e^{i\theta})| d\theta = 2\pi$. Show that f must be a constant function.

7. Suppose $f(z)$ is an analytic function on $|z| < 1$ such that $|f(z)| < 1$ for all $|z| < 1$ and $f(0) = \alpha \neq 0$. Show that $f(z) \neq 0$ for all $|z| < |\alpha|$.

8. Construct a meromorphic function $f(z)$ with a zero at $z = -1$, simple poles at $z = n$ for each positive integer n , and with the residue 1 at each pole. Justify that your function has the required properties. Are there any other functions satisfying these properties? If so, how many?