## COMPLEX ANALYSIS QUALIFYING EXAM

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Instructions: At the top of each page you use, write the secret code that you shared with Ana and label all problems appropriately. Complete all problems to get full credit. Start each problem on a new page, number the pages. Use only one side of each sheet. Clear and concise answers with good justification will improve your score. For integrations you need to introduce contours, branches, etc. explicitly!

1. Let $z, w$ be complex numbers lying in the first quadrant. Prove that

$$
|z|^{2}+|w|^{2} \leq|z+w|^{2} \leq 2\left(|z|^{2}+|w|^{2}\right) .
$$

2. Evaluate the definite integral

$$
I=\int_{0}^{\pi} \frac{\sin ^{2}(\theta) \mathrm{d} \theta}{10-6 \cos \theta}
$$

3. Expand the function

$$
f(z)=\frac{1+2 z^{2}}{z^{2}+z^{4}}
$$

into power series of $z$ in all areas of convergence.
4. Evaluate

$$
\operatorname{Res}_{z=0} \frac{\left(z^{6}-1\right)^{2}}{z^{5}\left(2 z^{4}-5 z^{2}+2\right)}
$$

5. Find all analytic bijections $f: \mathbb{C} \rightarrow \mathbb{C}$. Justify that there are no other analytic bijections besides those you found.
6. Suppose $f$ is an analytic function that maps the unit disk into itself and has zeros at $\frac{i}{2}$ and $-\frac{i}{2}$. Determine the largest possible value of $\left|f\left(\frac{1}{2}\right)\right|$ and give an example of a function $f$ that attains your upper bound.
7. If $f(z)$ is a one-to-one analytic function on the unit disk and $f(0)=0$, show that there is an analytic function $g(z)$ such that $g(z)^{2}=f\left(z^{2}\right)$ on the unit disk.
8. Show that the equation $\cos (\sqrt{z})=z$ has infinitely many solutions in $\mathbb{C}$.
