COMPLEX ANALYSIS QUALIFYING EXAM
JANUARY 7, 2022

Instructions: At the top of each page you use, write the secret code that you shared with Ana and label all problems. Complete all 8 problems to get full credit. Start each problem on a new page and **scan your solutions in numerical order**. Clear and concise answers with good justifications will improve your score.

In what follows, the unit disk $\{z \in \mathbb{C} : |z| < 1\}$ is denoted by $\mathbb{D}$.

1. Evaluate the improper integral
$$I = \int_0^{+\infty} \frac{x^2 \cos x}{(x^2+1)(x^2+9)} \mathrm{d}x.$$
Explain all steps carefully (show contours, introduce branches, etc.).

2. Find a conformal map from $\mathbb{D}$ to the open set $\Omega = \{z \in \mathbb{C} : 0 < \arg(z) < \frac{3\pi}{2}\}$.

3. Suppose $f(z)$ is an analytic function on the unit disk $\mathbb{D}$ and that there exists a sequence of points $z_n \in \mathbb{D}$ such that $|f(z_n)| \to \infty$ as $n \to \infty$. Prove that the radius of convergence of the power series for f about the origin is equal to 1.

4. Suppose that f is holomorphic on $\mathbb{C} \setminus \{0\}$ and that $f(n) = (-1)^n$ for each positive integer n. Prove that $\inf_{z \neq 0} |f(z)| = 0$.

5. Suppose α is a complex number with $|\alpha| > 5$ and let $p(z) = 3 + \alpha z + 2z^4$. How many zeros (including any multiplicity) does the polynomial $p(z)$ have in $\mathbb{D}$? Fully justify your answer.

6. Let $\Omega \subset \mathbb{C}$ be open and simply connected and let $u : \Omega \to \mathbb{R}$ be harmonic on Ω. Let
$$A := \left\{ z \in \Omega : \frac{\partial u}{\partial x}(z) = \frac{\partial u}{\partial y}(z) = 0 \right\}$$
be the set of points where the gradient of u vanishes. If A has an accumulation point in Ω, is it true that u is a constant? Fully justify your answer.

7. Find all entire functions $f(z)$ which are analytic and bounded on the upper half plane $U = \{z : \operatorname{Im} z > 0\}$ and real-valued on $\operatorname{Im} z = 0$ (the real axis).

8. Suppose f is a holomorphic function on the unit disk $\mathbb{D}$ and that $|f(z)| < 1$ for all $z \in \mathbb{D}$. Show that if f has a zero of order 2 at the origin, then $|f(z)| \leq |z|^2$ for all $z \in \mathbb{D}$.