COMPLEX ANALYSIS QUALIFYING EXAM

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Instructions: At the top of each page you use, write the secret code that you shared with Ana and label all problems appropriately. Complete all problems to get full credit. Start each problem on a new page, number the pages. Use only one side of each sheet. Clear and concise answers with good justification will improve your score.

1. Determine the image of the positively (counterclockwise) oriented square with vertices $1+i$, $-1+i$, $-1-i$, and $1-i$, under the map $w = e^z$. Sketch the (oriented) image of the square on the w-plane.

2. Let $f(z) = \dfrac{z^3 e^{1/z}}{z^2+1}$.
   (a) Find and classify all the singularities of $f(z)$ in the extended complex plane.
   (b) Does $|f(z)|$ diverge to infinity as $|z|$ goes to zero? Justify your answer.
   (c) Find the first two terms of the singular part of f in the domain $\{z \in \mathbb{C} : |z| < 1\}$.
   (d) Find the first four terms of the Laurent series of f in the domain $\{z \in \mathbb{C} : |z| > 1\}$.

3. Evaluate
$$\int_{-\infty}^{\infty} \frac{1}{1+x^4}\, dx$$

4. (a) Formulate Rouché's Theorem.
   (b) Determine the number of zeros of the function $f(z) = z^6 - 3z^5 + \cos z$ in the disk of radius 2 centered at the origin.

5. For $\alpha \in \mathbb{C}, |\alpha| < 1$, let
$$\phi_\alpha(z) = \frac{\alpha - z}{1 - \overline{\alpha} z} \quad \text{for} \quad z \in \mathcal{D} = D(0,1) = \{z \in \mathbb{C} \ : \ |z| < 1\} \ .$$
   (a) Find $\phi_\alpha \circ \phi_\alpha$.
   (b) Prove that ϕ is conformal (use your result in a).
   (c) Show that ϕ maps $\mathcal{D}$ onto $\mathcal{D}$.

6. Let $B_j, j = 0, 1, \ldots$ denote the sequence of Bernoulli numbers, defined by
$$\sum_{j=0}^{\infty} \frac{B_j}{j!}\, z^j = \frac{z}{e^z - 1} \quad \text{for} \quad 0 < |z| < \varepsilon \ . \tag{0.1}$$
   (a) Prove that $B_j = 0$ for j odd, $j \geq 3$.
   (b) Determine the radius of convergence of the power series in (0.1).

7. Let f be an analytic function on the (open) unit disk such that $|f(z)| \geq \sqrt{|z|}$ for all z on the unit disk. Show that $|f(z)| \geq 1$ on the unit disk.