COMPLEX ANALYSIS QUALIFYING EXAM
AUGUST 10, 2023
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Instructions: Hand in all of the following 8 problems. Start each problem on a new page, number the pages, and put only your code word (not your banner ID number) on each page. Submit your solutions arranged in numerical order. Clear and concise answers with good justification will improve your score.

1) Let
$$Q_1 = \Big\{ z = x + iy \ : \ x > 0 \quad \text{and} \quad y > 0 \Big\}$$
denote the first quadrant.
   a) Determine the image of Q_1 under the map $z \to e^{iz}$.
   b) Determine the image of Q_1 under the map $z \to \frac{1}{iz}$. Sketch the image.

2) Let $U \subset \mathbb{C}$ be an open set. Suppose $f : U \to \mathbb{C}$ is holomorphic and bounded. Fix $z_0 \in U$ and let r be the distance from z_0 to $\mathbb{C} \setminus U$, that is,
$$r = \inf\{|z_0 - w| : w \in \mathbb{C} \setminus U\}$$
Give a thorough proof that
$$\Big|\frac{d^k f}{dz^k}(z_0)\Big| \leq \frac{k!}{r^k} \sup_U |f|.$$

3) Let N be a fixed natural number. Suppose $\{p_j\}_{j=1}^{\infty}$ is a sequence of holomorphic polynomials and assume that the degree of each p_j is no more than N. Show that if $\{p_j\}_{j=1}^{\infty}$ converges uniformly on compact sets, then the limit function is a holomorphic polynomial of degree not exceeding N.

4) Evaluate the integral
$$\int_0^{2\pi} \frac{d\phi}{5 + 3\cos\phi} \ .$$
Hint: Transform to an integral along the unit circle.

5) (a) Formulate Rouché's Theorem.
   (b) How many zeros does the polynomial
$$p(z) = z^7 - 2z^5 + \frac{1}{2}$$
have outside the unit disk?

6) Let f be an entire function which is not a polynomial. Prove that for any $R > 0$
$$\inf_{|z|>R} |f(z)| = 0.$$

7) Let $\mathbb{D}$ denote the open unit disk and $\overline{\mathbb{D}}$ its closure. Suppose $f : \overline{\mathbb{D}} \to \mathbb{C}$ is continuous and that f is holomorphic on $\mathbb{D}$ with $|f(z)|$ is constant on $\partial\mathbb{D} = \{z : |z| = 1\}$. Prove that if f is not a constant function, then f has a zero inside of $\mathbb{D}$.

8) Let $p(z)$ be a nonconstant holomorphic polynomial. Show that the equation

$$e^z = p(z)$$

has infinitely many solutions in $\mathbb{C}$.