January 1994 ODE/PDE Exam

ID Number:

Part 1: Answer two of the five questions given below.
Ph.D students: At most one question can be of the M.A. type.
Only two questions will be graded. Write your chosen question numbers here:

(M.A.)1. Characterize the type of the critical points as a function of α for the system,

$$\begin{aligned} \frac{dx}{dt} &= (1-x^2)y, \\ \frac{dy}{dt} &= -x-\alpha y. \end{aligned}$$

(M.A.)2. Construct the total energy function, find and classify the equilibrium points, derive and solve the corresponding linear equations, and sketch a global phase plane portrait.

$$\frac{d^2x}{dt^2} + x + \frac{3}{x-4} = 0.$$

3.(a) Show that if (x_0, y_0) is a local minimum of $V(x,y)$, then $F(x,y) = V(x,y) - V(x_0,y_0)$ is a Liapunov function for the gradient flow,

$$\begin{aligned} \frac{dx}{dt} &= -\frac{\partial V(x,y)}{\partial x}, \\ \frac{dy}{dt} &= -\frac{\partial V(x,y)}{\partial y}, \end{aligned}$$

and that this guarantees stability of (x_0, y_0).
(b) Using the Liapunov stability theorem [**State it.**], can you determine if the origin is asymptotically stable or only that it is stable for the gradient flow,

$$\begin{aligned} \frac{dx}{dt} &= 4xy - 4x^3, \\ \frac{dy}{dt} &= 2x^2 - 2y, \end{aligned}$$

4. Let A be a 2×2 matrix with scalar entries. Then (by Cayley- Hamilton) A satisfies its characteristic equation, i.e. if $p(\lambda) = det(A - \lambda I)$, then $p(A) = 0$.
(a) Show that $Le^{tA} = 0$, where the differential operator L is defined by $L = p(\frac{d}{dt})$.
(b) Assume that the following is valid [It is.].

$$\begin{aligned} e^{tA} &= v(t)I + w(t)A, \\ \frac{d}{dt}e^{tA} &= \frac{d}{dt}v(t)I + \frac{d}{dt}w(t)A. \end{aligned}$$

Show that $Lv(t) = 0$, and $Lw(t) = 0$. Find $v(0), w(0), \frac{d}{dt}v(0), \frac{d}{dt}w(0)$.
(c) Use this information to compute e^{tA}, where

$$A = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}.$$

5. Consider the system

$$\begin{aligned} \frac{dx}{dt} &= x(y - y^2), \\ \frac{dy}{dt} &= x^2 - y. \end{aligned}$$

(a) Find the critical points and determine their stability.
(b) Show that if $x(0) > 0$, then $x(t) > 0$ for all $t \geq 0$.
(c) What is $\lim_{t\to\infty}(x(t), y(t))$ if $x(0) \geq 0$?
(Either sketch the trajectories of the phase plane portrait or argue based on your previous work.)

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→ **Part 2: Answer two of the five questions given below.**
Ph.D students: At most one question can be of the M.A. type.
Only two questions will be graded. Write your chosen question numbers here:

(M.A.)1(a). Find the earliest time at which a singularity develops for the equation

$$u_t + uu_x = 0, \quad -\infty < x < \infty, \quad 0 < t, \quad u(x,0) = \sin x.$$

(M.A.)1(b). Find the solution and domain of existence for

$$xu_x + (x+y)u_y = 1,$$

with the boundary condition, $u(1,y) = y, \quad 0 < y < 1$.

(M.A.)2. Find the solution to the wave equation,

$$y_{tt} = c^2 y_{xx}, \quad 0 < t, \quad 0 < x,$$

that satisfies the boundary condition, $y(0,t) = s(t)$, and the initial conditions,
$y(x,0) = 0, \quad y_t(x,0) = g(x)$.
Hint: Try $y(x,t) = f(x-ct) + h(x+ct)$.

3. Let $\Omega = G \times (0,T)$, where G is a bounded smooth domain in $\mathbf{R}^3$. Show that there is a unique solution to

$$U_t = \nabla \cdot (\kappa(\underline{x}) \nabla U) - c(\underline{x})U, \quad (\underline{x},t)\epsilon\Omega$$

with $U = f$ on $\partial\Omega$. The functions f, κ, c are smooth and c is positive.

4(a). Show that a fundamental solution of the differential operator, $\Delta + \frac{\omega^2}{c^2}$, Δ, the Laplacian in $\mathbf{R}^3$, is

$$G(\underline{x}, \underline{\zeta}) = -\frac{e^{-i\frac{\omega}{c}r}}{4\pi r},$$

where $r = |\, \underline{x} - \underline{\zeta} \,|$.
4(b). Using (a), find the solution to

$$(\Delta + \frac{\omega^2}{c^2})U(\underline{x}) = f(\underline{x}),$$

in a smooth bounded region Ω with $U = 0$ on $\partial\Omega$.

5. Prove: If the functions w_n are harmonic in a bounded domain G in $\mathbf{R}^2$, continuous in $\overline{G}$, then the sequence $\{w_n\}$ is uniformly convergent throughout $\overline{G}$, and the limit function is harmonic in G.