ODE/PDE Qualifying Exam

JANUARY 1995
~~Fall 199~~4

Name:________________

Part I: Answer 2 of the 4 questions given below.

1. Consider the system of ODE's

$$x' = x + y^2$$
$$y' = y - x^2$$

Find all the critical points and classify them. Also, solve the corresponding linearized equations around each critical point and sketch their local phase portrait.

2. Consider the initial value problem

$$x' = f(x, y), \quad x(0) = x_0,$$
$$y' = g(x, y), \quad y(0) = y_0$$

Let $C = \{(x(t), y(t)) | t \geq 0\}$ be the positive semi-orbit for the initial value problem. Assume that C is bounded and define C' as the set of points p in the plane for which there is a monotonic increasing sequence t_n with $t_n \to \infty$, and $(x(t_n), y(t_n)) \to p$ as $t_n \to \infty$.

(a) Show that $\lim_{t\to\infty} dist((x(t), y(t)), C') = 0$, where $dist(p, S)$ denotes the distance from the point p to the set S.

(b) Show that if $C' = \{p\}$, then $\lim_{t\to\infty}(x(t), y(t)) = p$. Does this imply that p is a stable equilibrium point? Prove it or give a counterexample.

3. Consider

$$x'' + 2\alpha x' + g(x) = 0$$

with $\alpha > 0$, $g(x)$ smooth and satisfying the conditions $xg(x) \geq 0$ in some open set containing the origin, and $g(0) = 0$. Show that $x(t) = 0$ is an asymptotically stable equilibrium by using the phase space function

$$f(x, x') = \frac{1}{2}(x' + \alpha x)^2 + \frac{1}{2}\alpha^2 x^2 + \int_0^x g(s) ds$$

4. (a) Let $r(t)$ be continuous for $0 \leq t - t_0 \leq \gamma$ and satisfy

$$0 \leq r(t) \leq \epsilon + \delta \int_{t_0}^{t} r(s) ds$$

for some non-negative constants ϵ and δ. Show that

$$0 \leq r(t) \leq \epsilon \, exp[\delta(t - t_0)]$$

for $0 \leq t - t_0 \leq \gamma$.

(b) Use the Gronwall inequality from (a) to prove uniqueness of solutions to the initial value problem

$$x' = f(x, t), \quad x(t_0) = x_0.$$

State the necessary assumptions on f.

Part II: Answer 2 of the 4 questions given below.

1. Solve the mixed initial-boundary-value problem

$$u_t + (1+2t)u_x = u, \quad x > 0, \ t > 0,$$

$$u(x,0) = 0, \quad x > 0,$$
$$u(0,t) = t, \quad t > 0$$

In particular, find the characteristics and sketch them in the first quadrant of the xt-plane.

2. Solve the mixed initial-boundary-value problem

$$u_t = u_{xx}, \quad 0 < x < 1, \ t > 0,$$

$$u(x,0) = 0, \quad 0 < x < 1,$$
$$u(0,t) = 0, \quad t > 0$$
$$u(1,t) = t, \quad t > 0$$

3. Consider the mixed initial-boundary-value problem

$$u_{tt} = u_{xx} - u_t, \quad 0 < x < \pi, \ t > 0$$

$$u(x,0) = f(x), \quad 0 < x < \pi,$$
$$u_t(x,0) = g(x), \quad 0 < x < \pi,$$
$$u_x(0,t) = 0, \quad t > 0,$$
$$u(\pi,t) = 0, \quad t > 0.$$

(a) Let

$$E(t) = \frac{1}{2}\int_0^{\pi} [(u_t)^2 + (u_x)^2] dx.$$

Show that $E(t) \leq E(0)$ for all $t > 0$.

(b) Solve the initial-boundary-value problem, if the initial data is given by $f(x) = \cos(\frac{x}{2})$, and $g(x) = 0$. Show that $E(t) \to 0$ as $t \to \infty$.

4. Consider the boundary-value problem for u

$$\Delta u = cu, \quad x \in \Omega,$$

$$u|_{\partial\Omega} = f, \quad x \in \partial\Omega,$$

where Ω is a bounded domain and c is a positive constant.

(a) Show that if $f \leq 0$ in $\partial\Omega$, then $u \leq 0$ in Ω, and if $f \geq 0$ in $\partial\Omega$, then $u \geq 0$ in Ω.

(b) Prove that if the solution of the boundary-value problem is unique.

(c) Is (a) still true if $c < 0$? Prove it or give a counterexample.

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