January 96

ODE/PDE Qualifying Exam

~~Fall 1995~~ *Name:*

Part I: Answer 2 of the 4 questions given below

1. Consider the system of ODE's

$$\frac{du}{dt} = -w^2 v$$
$$\frac{dv}{dt} = w^2 u + \frac{w^2}{2}$$
$$\frac{dw}{dt} = -\frac{wv}{2}$$

(a) Show that $u^2 + v^2 + w^2$ is a constant of the motion.

(b) Assuming that constant to be one. Use this fact to reduce the above system to a phase plane for any pair of variables $(u, v), (u, w), (v, w)$ that you prefer.

(c) Sketch the trajectories in the phase plane and find the critical points and classify them.

2. Consider the initial value problem

$$\frac{dx}{dt} = f(x, y), \quad x(0) = x_0$$
$$\frac{dy}{dt} = g(x, y), \quad y(0) = y_0$$

Let $C = \{(x(t), y(t)|t \geq 0\}$ be the positive semi-orbit for the initial value problem. Assume that C is bounded and define C' as the set of points p in the plane for which there is a monotonic increasing sequence t_n, with $t_n \to \infty$, and $(x(t_n), y(t_n)) \to p$ as $t_n \to \infty$.

(a) Show that $\lim_{t\to\infty} dist((x(t), y(t)), C') = 0$, where $dist(p, S)$ denotes the distance from the point p to the set S.

(b) Show that if $C' = p$, then $\lim_{t\to\infty}((x(t), y(t)) = p$. Does this imply that p is a stable equilibrium point? Prove or give a counterexample.

3. Consider the system

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = -2\alpha^2 y + g(x)$$

with $g(x)$ smooth and satisfying the condition $xg(x) < 0$, $x \neq 0$ in some open set containing $x = 0$ and $g(0) = 0$. Show that $(0,0)$ is an asymptotically stable equilibrium, by using the phase space function

$$f(x,y) = \frac{1}{2}(y + \alpha^2 x)^2 + \frac{1}{2}\alpha^4 x^2 - \int_0^x g(s)ds$$

$\alpha \neq 0$

4. (a) Let $r(t)$ be continuous for $0 \leq t - t_0 \leq \gamma$ and satisfy

$$0 \leq r(t) \leq \epsilon + \delta \int_{t_0}^{t} r(s)ds$$

for some non-negative constants ϵ, δ. Show that

$$0 \leq r(t) \leq \epsilon exp[\delta(t - t_0)]$$

for $0 \leq t - t_0 \leq \gamma$.

(b) Use Gronwall's inequality from (a) to prove uniqueness of solutions to the initial value problem

$$\frac{dx}{dt} = f(x,t), \quad x(t_0) = x_0$$

State the necessary assumptions on f.

Part II: Answer 2 of the 4 questions given below

1. Solve the value problem

$$tu_t + (x+t)u_x = 1$$
$$u(1,x) = x, \ \ for \ 0 < x < 1$$

and describe the region over which the solution is uniquely determined.

2. Consider the wave equation in one dimension

$$u_{tt} - u_{xx} = 0$$

with initial conditions $u(x,0) = f(x)$, $u_t(x,0) = g(x)$, $-\infty < x < \infty$.

(a) Suppose f and g are identically zero outside the interval $[-1,1]$. In what region in $(-\infty,\infty)) \times [0,\infty)$ can you ensure that the solution u of the Cauchy problem is identically zero.

(b) Is there a similar result to the previous problem for the heat equation? (Hint: Think of the fundamental solution).

3. Consider the diffusion equation

$$u_t = A^2(x^2 u_x)_x$$
$$1 < x < 2, \ \ t > 0$$

With initial and boundary conditions:

$$u(x,0) = g(x), \ \ u(1,t) = u(2,t) = 0$$

(a) Show that if $u(x,t) = \frac{1}{\sqrt{x}} f(t,y)$, where $y = ln(x)$. Then f satisfies a linear second order PDE, with constant coefficients.

(b) From the equation you derived in (a), use separation of variables to find the eigenfunctions and eigenvalues of this problem

4. Given that

$$u(x,y) = \frac{1-x^2-y^2}{1-2x+x^2+y^2} \tag{6}$$

is harmonic (ie. satisfies Laplaces equation) for $x^2+y^2<1$ Then:

(a) is it harmonic for $x^2+y^2<a,\quad a>1$?. Explain

(b) Prove that $\frac{a-r}{a+r} \le u(x,y) \le \frac{a+r}{a-r}$ for $r<a$ is true for all $a<1$.
(here $r^2=x^2+y^2$)

(c) Prove that $\frac{1-a}{1+a} \le u(x,y) \le \frac{1+a}{1-a}$ for $r \le a$, for all $a<1$.

Hint: Go to polar coordinates throughout this exercise