August 96

Differential Equations Qualifying Exam

Do problems 1 and 2 along with any two of problems 3-5.

1. Consider the initial value problem:

$$y' = t^2 + y^2, \quad y(0) = 1.$$

a. State the existence, uniqueness, continuation theorem as it relates to this problem. Prove uniqueness.

b. Solve the initial value problems:

$$y' = y^2 \quad \text{and} \quad y' = 1 + y^2,$$

for $y(0) = 1$ and $t \geq 0$. Find the maximal (right) interval of existence for these problems. (Hint: $\frac{d}{dx}\tan^{-1} x = (1 + x^2)^{-1}$.)

c. Prove that the maximal right interval of existence in (a.) lies between the two in (b.).

2. Consider the differential equation:

$$\frac{d^2x}{dt^2} + a\epsilon\left(\frac{dx}{dt}\right)^3 + x - \epsilon^3 x^3 = 0, \quad \epsilon > 0,$$

and its equivalent system

$$\begin{aligned}\frac{dx}{dt} &= y,\\ \frac{dy}{dt} &= -x + \epsilon^3 x^3 - a\epsilon y^3.\end{aligned}$$

a. Sketch the phase plane portrait for $a = 0$.

b. Find the equilibrium solutions (critical points), linearize about them, and discuss the phase plane portraits of the linearized systems. (You do not need to solve them.)

c. Use the theory of almost linear systems to discuss the behavior of the nonlinear system near the critical points. If the case is indeterminant and $a \geq 0$ argue by other means about the stability; e.g. consider the energy function $\frac{1}{2}\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}x^2 - \frac{1}{4}\epsilon^2 x^4$.

d. Consider the initial value problem $x(0) = 1$, $x'(0) = 0$ and use regular perturbation theory to find $x_1(t)$ and $x_2(t)$ where

$$x(t) = x_1(t) + \epsilon x_2(t) + O(\epsilon^2).$$

On what time interval is your solution valid? Are your results consistent with your analysis in (c.)?

3. Consider the space-periodic initial value problem:

$$u_t = \begin{pmatrix} 0 & \alpha \\ \beta & \gamma \end{pmatrix} u_{xx}, \quad t > 0,$$

$$u(x,0) = u_0(x), \quad u(x+2\pi, t) = u(x,t).$$

a. Formally solve by Fourier series.

b. Give necessary conditions on the real parameters α, β and γ for the well-posedness of the problem.

4. Solve the initial value problem:

$$u_t + u^2 u_x + u = 0, \quad u(x,0) = A \sin x,$$

implicitly using the method of characteristics. Find the time and location(s) of shock formation. For what, if any, $A > 0$ do we have a smooth solution for all time?

5. Suppose $f(u)$ is a smooth function satisfying:

$$f(u) < 0, \text{ if } u > 0; \quad f(u) > 0, \text{ if } u < 0.$$

Show that $u = 0$ is the only solution of:

$$Lu + f(u) = 0, \quad x \in \Omega; \quad u = 0, \quad x \in \partial\Omega,$$

where Ω is a bounded domain with smooth boundary and L is the elliptic operator with smooth coefficients defined by:

$$Lu \equiv \nabla \cdot A(x) \nabla u,$$

with A a uniformly positive-definite symmetric matrix.