ODE/PDE EXAM – January 1997

PART I - ODE: Do all three problems.

1. Solve the IVP $y' = y^2$, $y(0) = a$, where a is real, and discuss the maximal interval of existence. Define the Picard iterates for this problem. Do you expect the Picard iterates to converge on the maximal interval? Is your solution unique? Why?

2. a) Consider $\dot{x} = f(x)$. Suppose $x_p(t)$ is a periodic solution and let $x(t) = x_p(t) + u(t)$. Find the equation for u. Show that linearization about the periodic orbit leads to a linear system with periodic coefficients given by

$$\dot{v} = A(t)v\,,\quad A(t) = f'(x_p(t)). \tag{1}$$

b) Let $f(x) = \begin{pmatrix} x_1 - 4x_2 - x_1(\frac{1}{4}x_1^2 + x_2^2) \\ x_1 + x_2 - x_2(\frac{1}{4}x_1^2 + x_2^2) \end{pmatrix}$.

Show that $x_p(t) = \begin{pmatrix} 2\cos 2t \\ \sin 2t \end{pmatrix}$ is a periodic solution. Find $A(t)$ and show that

$$\Phi(t) = \begin{pmatrix} e^{-2t}\cos 2t & -2\sin 2t \\ \frac{1}{2}e^{-2t}\sin 2t & \cos 2t \end{pmatrix}$$

is a fundamental solution matrix for (1). Does $\Phi(t) = \exp \int_0^t A(s)ds$? Explain.

c) State the Floquet Theorem for (1) and write $\Phi(t)$ above in its Floquet form. What are the characteristic exponents and the characteristic multipliers?

3. Consider the IVP

$$\dot{x} = \varepsilon f(x,t)\,,\quad x(0) = x_0 \tag{1}$$

where f is smooth, bounded, globally x-Lipschitz and 2π periodic in t.

Define $\bar{f}(x) = \frac{1}{2\pi}\int_0^{2\pi} f(x,t)dt$ and the IVP

$$\dot{v} = \varepsilon \bar{f}(v),\quad v(0) = x_0. \tag{2}$$

a) Show that there exists a transformation $v \to w$ via

$$w = v + \varepsilon P(v,t)\,,\ P(v,0) = 0\,,\ P(v,t+2\pi) = P(v,t) \tag{3}$$

which transforms (2) into the IVP

$$\dot{w} = \varepsilon f(w,t) + \varepsilon^2 h(v,t,\varepsilon),\ w(0) = x_0\,. \tag{4}$$

That is, find P and h which do the job and show that $|h|$ is bounded. Are there any restrictions on ε?

b) Let $e(t) = x(t) - w(t)$ and show that

$$|e(t)| \leq \varepsilon L \int_0^t |e(s)|ds + \varepsilon^2 Mt \tag{5}$$

where L is the (global) Lipschitz constant for f and M is the bound for h.

c) Assuming that the solutions of (1), (2) and (3) exist on $[0, T/\varepsilon]$, prove that $|x(t) - v(t)| \leq C(T)\varepsilon$ for $0 \leq t \leq T/\varepsilon$. Do you need to restrict ε to get this result? Hint: Use (3), (5) and the Gronwall inequality.

d) Assume that the solution of (2), and thus (3), exists on $[0, T/\varepsilon]$. Show that the solution of (1) also exists on $[0, T/\varepsilon]$ by possibly restricting ε. State the continuation theorem and use it in your proof.

PART II - PDE: Work out any 2 of the 4 problems given below.

1. Consider the equation

$$u_t + xu_x = 0\,, \quad x \in \mathbb{R}, \quad t \geq 0$$

with initial condition

$$u(x, 0) = \cos x\,, \quad x \in \mathbb{R}.$$

a) Sketch the family of characteristic lines in the (x, t) plane (along each characteristic line the solution is constant), and determine the solution.

b) With the same initial condition, solve

$$u_t + xu_x = 1\,, \quad x \in \mathbb{R}, \quad t \geq 0\,.$$

c) With the same initial condition, solve

$$u_t + xu_x = u\,, \quad x \in \mathbb{R}, \quad t \geq 0\,.$$

2. Consider the IBVP

$$\begin{aligned} u_t = u_{xx}\,, \quad 0 \leq x \leq 1 \quad t \geq 0\,, \\ u(0, t) = u_x(1, t) = 0\,, \quad t \geq 0\,, \\ u(x, 0) = f(x)\,, \quad 0 \leq x \leq 1\,, \end{aligned}$$

where $f(x)$ is a smooth function.

a) For which functions $f(x)$ can you obtain a solution

$$u(x,t) = \varphi(t) f(x)$$

in separated variables? What is the solution?

b) Solve the IBVP for $f(x) \equiv 1$ in series form. (There is an incompatability of boundary and initial data, which you may ignore.)

3. Consider the IVPs

(a) $u_t = u_{xxx} + u_{xx}\,, \quad x \in \mathbb{R}, \quad t \geq 0,$

(b) $u_t = u_{xxx} - u_{xx}\,, \quad x \in \mathbb{R}, \quad t \geq 0,$ with initial condition

$$u(x,0) = f(x)\,, \quad x \in \mathbb{R}.$$

We assume that $f(x)$ and $u(x,t)$ are 2π-periodic in x.

a) Assuming that $f(x)$ is a trigonometric polynomial,

$$f(x) = \sum_{k=-n}^{n} c_k e^{ikx}\,, \quad c_k \in \mathbb{C},$$

obtain solutions $u(x,t)$ using Fourier expansion in x.

b) One of the IVPs is well-posed, the other ill-posed. Define these concepts and apply them to the two examples.

4. a) The function

$$G(x) = \frac{1}{4\pi|x|}\,, \quad |x|^2 = x_1^2 + x_2^2 + x_3^2\,, \; x \in \mathbb{R}^3\,,$$

is a fundamental solution of

$$-\Delta u(x) = 0\,, x \in \mathbb{R}^3\,.$$

Explain what this means.

b) Is $G(x)$ integrable over $\mathbb{R}^3$?
Is $G(x)$integrable over each compact subset of $\mathbb{R}^3$?

c) Given a continuous function $f : \mathbb{R}^3 \to \mathbb{R}$ with compact support, write a solution of

$$-\Delta u(x) = f(x)\,, \quad x \in \mathbb{R}^3\,,$$

as a convolution integral with $G(x)$. Show that the solution decays to 0 as $|x| \to \infty$.