QUALIFYING EXAM
ODE / PDE
Fall 1997

Department of Mathematics and Statistics
University of New Mexico

Note: The problems on this exam cover material in the topics specified in the ODE and PDE qualifying exam descriptions available from the department. You are responsible for all of this material whether or not the topics were covered in your courses.

Instructions: Clearly presented, concise (and correct!) answers are given more credit! Those trying to qualify for the Ph.D. should solve all required problems. Those trying to complete a Master's degree should solve completely a substantial number of the required problems.

ODE Problems

Solve 3 of the 4 problems.

1. Consider the system

$$\frac{dx}{dt} = x\,\left(\lambda - (x^2 + (1+\epsilon^2)y^2)\right) + \omega\, y\,, \quad \frac{dy}{dt} = -\omega\, x + y\,\left(\lambda - (x^2 + (1+\epsilon^2)y^2)\right)\,. \quad (1)$$

   Show that the system has a stable limit cycle for $\lambda, \epsilon > 0$.

2. Prove Gronwall's Lemma:

   Suppose that $g(t)$ is a continuous real-valued function that satisfies $g(t) \geq 0$ and

$$g(t) \leq C + K\int_0^t g(s)\,ds \tag{2}$$

   for all $t \in [0, a]$, where C and K are positive constants. Show that for all $t \in [0, a]$

$$g(t) \leq C\,e^{K\,t}\,. \tag{3}$$

   Use this lemma to prove that the initial-value problem

$$\frac{dy}{dt}(t) = e^{\sin(t)}\,y(t)\,, \quad y(0) = y_0 \tag{4}$$

   with y_0 given, has an infinite interval of existence for its solutions.

3. A fundamental matrix Φ for the periodic system of differential equations

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & h(t) \end{pmatrix}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \tag{5}$$

   where

$$h(t) = \frac{\cos(t) + \sin(t)}{2 + \sin(t) - \cos(t)} \tag{6}$$

   is given by

$$\Phi(t) = \begin{pmatrix} -2 - \sin(t) & e^t \\ 2 + \sin(t) - \cos(t) & 0 \end{pmatrix}, \tag{7}$$

   (a) Show that in fact $\Phi(t)$ is a fundamental matrix.

   (b) State the Floquet Theorem and then compute the Floquet exponents for the above problem. Discuss the stability of the solutions of the initial-value problem for this system.

4. For the second-order ordinary differential equation

$$\frac{d^2x}{dt^2} = \nu - x^2\,, \tag{8}$$

   determine all critical points and their stability for all real values of $\nu \neq 0$.

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PDE Problems

Solve 3 of the 4 problems.

1. Find $u = u(x,t)$, $0 \le x \le 1$, $0 \le t < \infty$, that satisfies the initial-boundary value problem given by the PDE
$$\frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t) \tag{9}$$
on the interior of the region and the initial and boundary conditions
$$u(x,0) = f(x) \in C^1([0,1]), \quad \frac{\partial u}{\partial x}(0,t) = 0, \quad \frac{\partial u}{\partial x}(1,t) = 0. \tag{10}$$
   (a) Under what condition(s) does $u(x,t) \to 0$ as $t \to \infty$?

   (b) Define
$$M(t) = \int_0^1 u^2(x,t)dt \quad . \tag{11}$$
   Derive an explicit expression for $M(t)$ in terms of the Fourier coefficients f_k of the initial data $f(x)$. Show that $M(t) < \infty$. Find
$$M_\infty = \lim_{t\to\infty} M(t) \quad . \tag{12}$$
   Under what condition(s) is $M_\infty = 0$?

   (c) Suppose that the initial data $f(x)$
$$\sum_0^\infty |f_k| = N < \infty \quad , \tag{13}$$
   with f_k the k-th Fourier coefficient of $f(x)$. Find a function $g(t)$ such that
$$\max_{x\in[0,1]} |u(x,t)| \le g(t)N \quad . \tag{14}$$
   Under what condition(s) is
$$\lim_{t\to\infty} g(t) = 0 \quad ? \tag{15}$$

2. Consider the following system of partial differential equations for $u = u(\vec{x},t)$ and $\vec{v} = \vec{v}(\vec{x},t)$ with $-\infty < t < \infty$ and all $\vec{x} \in R^3$:
$$\frac{\partial u}{\partial t} = \nabla\cdot\vec{v}, \quad \frac{\partial \vec{v}}{\partial t} = A\,\nabla u \tag{16}$$
where A is a constant, symmetric,positive definite matrix, $\nabla\cdot$ is the divergence, and ∇ is the gradient. When A is the identity matrix and $\vec{x} = (x,y,z)$, the energy for this equation is
$$E(t) = \int_{R^3} \left\{ \left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 \right\} dx\,dy\,dz\,. \tag{17}$$

(a) Using vector notation, define the energy for Equation (16) and then prove that your definition is correct, i.e. the enrgy stays constant in time for compactly supported data.

(b) Can you infer from the energy that all solutions of the system are bounded?

(c) Eliminate $\vec{v}$ from the system of equation to obtain a second-order equation for u. Check that $u(\vec{x}, t) \equiv t$ is a solution that is not bounded. What are the corresponding solutions of the system (16)?

3. Consider the first order PDE

$$\frac{\partial u}{\partial x} + 2\sqrt{y}\frac{\partial u}{\partial y} = 0, \quad y \geq 0 \quad \text{with } u \text{ given on the curve} \quad x^2 = y. \tag{18}$$

(a) Find the characteristics.

(b) Determine the region in which the function $u(x, y)$ can be uniquely determined for this problem.

(c) What choice of boundary values would make the region in (b) as large as possible?

4. Assume that $u = u(x, t)$, $0 \leq x \leq 1$, $0 \leq t < \infty$, satisfies the initial-boundary value problem given by the PDE

$$\frac{\partial u}{\partial t}(x, t) = \frac{\partial^2 u}{\partial x^2}(x, t) \tag{19}$$

on the interior of the region and the initial and boundary conditions

$$u(x, 0) = f(x), \quad u(0, t) = \alpha(t), \quad u(1, t) = \beta(t). \tag{20}$$

(a) Show that $u(x, t)$ cannot have a maximum where $\partial^2 u/\partial x^2 < 0$ in the interior of the region in (x, t) space with $t > 0$ and $0 < x < 1$,

(b) State the strong maximum/minimum principle for the previous initial-boundary value problem. (This means the strongest possible result about interior maxima and minima.)

(c) Using a maximum/minimum principle show that if $f(x) \geq 0$, $\alpha(t) \geq 0$, and $\beta(t) \geq 0$, then $u(x, t) \geq 0$.