ODE/PDE Preliminary Exam, Fall 1998

Note: All the questions must be answered.

PART 1: ODEs

1. Consider $\mathbf{x}' = A(t)\mathbf{x}$, where $A(t)$ is a continuous 2×2 matrix with $A(t+1) = A(t)$. Let $\Phi(t)$ be the fundamental matrix solution, and assume that trace $[\Phi(1)] = 2\alpha$, where α is a real constant. Show that if $\int_0^1 \text{trace}\,[A(t)]\,dt = -\ln 2$ and if $|\alpha| < 3/4$, then all solutions approach $\mathbf{x} = \mathbf{0}$ as $t \to \infty$.

2. Consider the ODE

$$\begin{aligned} x' &= \lambda x - y - x(4x^2 + 7y^2) \\ y' &= x + \lambda y - y(4x^2 + 7y^2). \end{aligned}$$

Show that if $\lambda > 0$, then there is at least one nonconstant periodic solution. (Hint: Write the ODE in polar coordinates)

3. Consider the linear ODE

$$\mathbf{x}' = A\mathbf{x} + B(t)\mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0.$$

Suppose that all of the eigenvalues of the $n \times n$ matrix A have negative real part, and that

$$\int_0^\infty \|B(t)\|\,dt < \infty.$$

Show that

$$\lim_{t\to\infty} |\mathbf{x}(t)| = 0.$$

(Hint: Derive an inequality for $|\mathbf{x}(t)|$)

4. Show that $\mathbf{x} = \mathbf{0}$ is an asymptotically stable equilibrium point for the system

$$\begin{aligned} x_1' &= -x_2 - x_1x_2^2 - x_1^3 \\ x_2' &= x_1 - x_1^2x_2 - x_2^3. \end{aligned}$$

(Hint: The function $V(x_1, x_2) = x_1^2 + x_2^2$ may prove to be useful)

PART 2: PDEs

1 Show that the solution of the nonlinear equation

$$u_x + u_y = u^2 \tag{1}$$

satisfying $u = t$ along the curve $x = t$, $y = -t$, becomes infinite along the hyperbola $x^2 - y^2 = 4$

2 Solve

$$u_{tt} - c^2 u_{xx} = 0$$

in $x > 0$, $t > 0$, with initial conditions

$$u(x,0) = f(x), \quad u_t(x,0) = g(x)$$

and boundary condition $u_t(0,t) = \alpha u_x(0,t)$, where α is a constant.

(b) Show that in general, no solution exists when $\alpha = -c$

3 Prove that the solution of Poisson equation,

$$\Delta u = f, \quad \mathbf{x} \epsilon \Omega \tag{2}$$

with boundary condition

$$\frac{\partial u}{\partial n} + \alpha u = g, \quad \mathbf{x} \epsilon \partial\Omega$$

is unique when $\alpha > 0$. (Here $\frac{\partial u}{\partial n}$ refers to the normal derivative).

(b) For $\alpha = 0$ show that $\int_\Omega f d\mathbf{x} = \int_{\partial\Omega} g dS$ is a necessary condition for the solution of the Neumann problem to exist.