## ODE Exam, Winter 1999

1. Consider $x^{\prime \prime}+x^{3}+x^{2}-2 x=0$.
a. Find the critical points (equilibrium solutions).
b. Linearize about the critical points and solve the linearized equations. Discuss the stability of the critical points based on the linearization. If the critical point is a center find the period of the linearized equation.
c. Construct the energy function and use it to sketch the phase plane portrait. Discuss the stability of the critical points based on your result.
d. For what initial conditions $x(0)=x_{0}, x^{\prime}(0)=y_{0}$ are the solutions of the ode periodic. What happens to the period as a scparatrix is approached.
2. Consider $\mathbf{x}^{\prime}=\Lambda(t) \mathbf{x}$, where

$$
A(t)=\left[\begin{array}{cc}
-1+(3 / 2) \cos ^{2} t & 1-(3 / 4) \sin 2 t \\
-1-(3 / 4) \sin 2 t & -1+(3 / 2) \sin ^{2} t
\end{array}\right] .
$$

a. Verify that the vector $(-\cos t, \sin t) e^{t / 2}$ is a solution of the linear system.
b. Find the Floquet (characteristic) multipliers associated with the period $2 \pi$ of $A$. We do not recommend trying to find a second linearly independent solution.
3. Let $f: \mathbf{R} \times \mathbf{R} \rightarrow \mathbf{R}$ be a $C^{1}$ function with $\left|f_{x}\right| \leq 1$ for all $(t, x) \in \mathbf{R} \times \mathbf{R}$. Let $a(\lambda) \in C^{1}(\mathbf{R})$ satisfy $\left|a^{\prime}(\lambda)\right| \leq 1$ for all $\lambda$. Denote the solution of the initial value problem

$$
x^{\prime}=f(t, x), \quad x(0)=a(\lambda)
$$

by $x_{\lambda}(t)$.
a. Show that for $t$ in the domain of both $x_{\lambda}(t)$ and $x_{\gamma}(t)$

$$
\left|x_{\lambda}(t)-x_{\gamma}(t)\right| \leq|\lambda-\gamma|+\int_{0}^{t}\left|x_{\lambda}(s)-x_{\gamma}(s)\right| d s
$$

b. Show that whenever it exists,

$$
\left|\frac{\partial}{\partial \lambda} x_{\lambda}(1)\right| \leq 3 .
$$

1) Let $f(r)$ denote a smooth function defined for all real $r>0$ and set

$$
u(x, y)=f\left(\sqrt{x^{2}+y^{2}}\right), \quad(x, y) \neq(0,0)
$$

a) Show that

$$
\Delta u=f^{\prime \prime}+\frac{1}{r} f^{\prime}
$$

where $\Delta u=u_{x x}+u_{y y}$.
b) Use a) to determine all functions $u(x, y)$ which depend only on $r=\sqrt{x^{2}+y^{2}}$ and which satisfy $\Delta u=0$ in the whole plane except at $(x, y)=(0,0)$.
2) Consider the nonlinear equation

$$
u_{t}+u u_{x}=0, \quad x \in \mathbb{R}, \quad t \geq 0
$$

with initial condition

$$
u(x, 0)=\sin x
$$

a) Explain how characteristics can be used to construct a smooth solution in some time interval $0 \leq t \leq t_{0}, t_{0}>0$.
b) Explain why a smooth solution does not exist beyond $t=1$.
c) Consider

$$
v_{t}+v v_{x}=-v, \quad x \in \mathbb{R}, \quad t \geq 0,
$$

with the same initial condition,

$$
v(x, 0)=\sin x
$$

Can one construct a smooth solution for all $t \geq 0$ ?
3) Consider the equations

$$
u_{t}=u_{x x}, \quad x \in \mathbb{R}, \quad t \geq 0
$$

and

$$
v_{t}=-v_{x x}, \quad x \in \mathbb{R}, \quad t \geq 0
$$

with initial condition

$$
u(x, 0)=v(x, 0)=\sin (n x)
$$

where $n$ is an integer.
a) Determine solutions, $u(x, t)$ and $v(x, t)$, which are $2 \pi$-periodic in $x$.
b) Use the two examples to discuss the concepts of well-posedness and ill-posedness.

