ODE/PDE Qualifying Examination

August 1999 SS # : ______________

Directions:

Start each question on a new sheet of paper. Write only on one side of each sheet of paper. Number the pages and *write your SS #* in each page. Indicate clearly below which questions you are attempting.

Questions:

ODE part: Answer *any two* of problems 1-3 and *one* of problems 4-5.

1. (a) Solve $x' = 2\frac{x^2}{t^2}$, with initial condition $x(1) = 1$. Find the maximum time $T > 1$, so that the solution is defined on $[1, T)$.

   (b) Find all the solutions of $x' = 2\frac{x^2}{t^2}$, with initial condition $x(0) = 0$. Does your answer contradict the uniqueness theorem? Explain.

2. Consider the nonlinear ODE system

$$\begin{aligned} x' &= x - 1 - y \\ y' &= -y - (x-1)^3 \end{aligned} \tag{1}$$

   (a) Show that (1) has only one equilibrium point $\bar{x}_* = (x_*, y_*)$. Find the linearized equations around $\bar{x}_* = (x_*, y_*)$ and discuss the nonlinear stability of the equilibrium point.

   (b) Write the linearized equations in matrix form, $\bar{x}\,' = A\bar{x}$, and find the exponential e^{At} of the matrix A.

3. Consider the nonlinear second order equation

$$x'' = -x + x^2 \tag{2}$$

   (a) Find the total energy $E = E(x, x')$ for this equation and show that E is conserved.

   (b) Use the total energy E to sketch the solutions of (2) in the phase plane (x, x').

   (c) Identify the equilibrium points of (2) and determine whether they are stable or unstable.

   (d) For what values of A is the solution of (2) with initial conditions $x(0) = A$, $x'(0) = 0$ periodic?

4. (a) Prove Gronwall's inequality: If $u(t) \geq 0$ is a continuous function on $0 \leq t \leq T$ and satisfies

$$u(t) \leq u_0 + \int_0^t K(s)u(s)\,ds, \quad 0 \leq t \leq T$$

   with $u_0 \geq 0$ and $K(t)$ a non-negative continuous function on $0 \leq t \leq T$, then $u(t)$ satisfies

$$u(t) \leq u_0 e^{\int_0^t K(s)\,ds}, \quad 0 \leq t \leq T.$$

(b) Use Gronwall's inequality to show that there is a unique solution $u(t)$ of the initial value problem

$$\begin{aligned} u' &= t\sin(u+t), \quad 0 < t < \infty \\ u(0) &= 1 \end{aligned} \tag{3}$$

You can assume that any solution $u(t)$ of (3) is defined on $0 \le t < \infty$.

5. Consider the nonlinear ODE system

$$\begin{aligned} x' &= -x - y + x(x^2+y^2), \quad x(0) = x_0, \\ y' &= x - y + y(x^2+y^2), \quad y(0) = y_0. \end{aligned} \tag{4}$$

(a) Show that $V(x,y) = x^2 + y^2$ is a Liapunov function in a neighborhood of $(x,y) = (0,0)$.

(b) Utilize $V(x,y)$ to prove directly (that is, without quoting Liapunov's theorem) that $(0,0)$ is an asymptotically stable equilibrium point.

(c) Find $R > 0$ so that if the initial data (x_0, y_0) is in the disc of center (0.0) and radius R, then the solution $(x(t), y(t))$ of (4) converges to $(0,0)$ as $t \to \infty$.

PDE part: Answer *any two* of problems 1-3 and *one* of problems 4-5.

1. Consider the initial-boundary-value problem

$$\begin{aligned} &u_t = u_{xx} + e^{-t}\sin\pi x - \sin 3\pi x \\ &u(0,t) = 0 \\ &u(1,t) = 0 \\ &u(x,0) = \sin\pi x \end{aligned}$$

Find the solution and discuss its behavior as $t \to \infty$.

2. Solve Laplace's equation on the disk, that is, solve the boundary-value problem

$$\begin{aligned} &u_{rr} + \tfrac{1}{r}u_r + \tfrac{1}{r^2}u_{\theta\theta} = 0, \quad 0 < r < 1, \quad 0 \le \theta \le 2\pi, \\ &u(1,\theta) = f(\theta), \quad 0 \le \theta \le 2\pi, \\ &u(r,\theta) \text{ bounded around } r = 0. \end{aligned}$$

3. (a) Solve the initial-value problem

$$\begin{aligned} &u_{tt} - u_{xx} = 0, \quad -\infty < x < \infty, \quad 0 < t < \infty \\ &u(x,0) = \begin{cases} 1 & x \in [0,1] \\ 0 & \text{otherwise} \end{cases} \\ &u_t(x,0) = 0 \end{aligned}$$

(b) For fixed $t > 0$, identify the x-region where $u = 0$ and where $u \neq 0$.

4. Consider the initial-value problem

$$\rho_t - (y\rho)_y = \rho_t - y\rho_y - \rho = 0, \quad -\infty < y < \infty, \quad 0 < t < \infty,$$
$$\rho(y,0) = \rho_0(y).$$

(a) Solve the initial-value problem and sketch the characteristics in the y-t plane.

(b) Use your solution in a) with the initial data

$$\rho_0(y) = \begin{cases} 1 & y \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

to compute $\int_{-\infty}^{\infty} \rho(y,t)\, dy$.

(c) Can you verify your result in b) directly from the PDE? Explain.

5. Consider the boundary-value problem

$$u_{xx} + u_{yy} = 0, \quad -\infty < x < \infty, \quad 0 < y < \infty \tag{1}$$
$$u(x,0) = f(x) \tag{2}$$

and its solution

$$u(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y}{(x-\xi)^2 + y^2} f(\xi)\, d\xi \tag{3}$$

(a) Show that $u(x,y)$ in (3) satisfies (1).

(b) Plug $y = 0$ into the integrand in (3). Does it satisfy the boundary condition in (2)? Explain. Your explanation could be in terms of delta sequences.

(c) Find $u(x,y)$ explicitly for $f(x)$ given by

$$f(x) = \begin{cases} 1 & x < 0 \\ 0 & x > 0 \end{cases}$$

and prove that the boundary condition in (2) is satisfied.