ODE Exam
January 2000

Instructions: Answer one of problems 1 and 2, and one of problems 3 and 4. Circle the two problems you want graded.

1. Consider the nonlinear oscillator (NO) $\ddot{x} + g(x) = 0$ or in system form $\dot{x} = y, \dot{y} = -g(x)$.

   (a) Define a critical point (CP) (x_c, y_c) in terms of g.

   (b) Let (x_c, y_c) be a CP. Linearize the NO equation about the CP and discuss linearized stability in terms of g. What can you conclude at this stage about (nonlinear) stability?

   (c) Show that $E(x, y) = \frac{1}{2}y^2 + G(x)$ where $G'(x) = g(x)$ is constant along any solution of the NO equation.

   (d) Construct the phase plane portrait for the NO with $g(x) = x - \frac{1}{a-x}$ for $x < a$. Discuss the qualitative behavior of solutions for all IC's (x_0, y_0) and $x_0 < a$. Include the CPs and their (nonlinear) stability in your discussion.

2. Consider $\dot{x} = Ax \quad (*)$, where A is a constant $n \times n$ matrix.

   (a) Define e^{At} and show that the solution of the IVP $x(0) = x_0$ is $x(t) = e^{At}x_0$.

   (b) Show that $e^{A(t+\tau)} = e^{At}e^{A\tau}$ by using a uniqueness theorem for solutions of $*$.

   (c) Find e^{At} for $A = \begin{pmatrix} 0 & 1 \\ 6 & 1 \end{pmatrix}$. Verify that $\left(e^{A(t+\tau)}\right)_{11} = \left(e^{At}e^{A\tau}\right)_{11}$ from your explicit formula.

   (d) Find the solution of $\dot{x} = Ax + f(t)$ in terms of e^{At} and $f(t)$.

3. Consider the IVP $\dot{x} = f(x,t), x(0) = z$, and let $\varphi(t,z)$ be the solution with $\varphi(0,z) = z$.

   (a) Define continuity of φ in z for fixed t.

   (b) Specify general conditions on f so that solutions of the IVP exist uniquely on some open interval containing $t = 0$ and are continuous in z for fixed t.

   (c) Prove the continuity in z for fixed t for your conditions in (b). Use a form of the Gronwall inequality in your proof.

   (d) Prove the version of the Gronwall inequality you used in (c).

4. Suppose $x_p(t)$ is a T-periodic solution of $\dot{x} = f(x), x \in \mathbf{R}^n$.

   (a) Define u by $x = x_p + u$ and find the linearized equation for u, i.e., find $A(t)$ in terms of f such that the linearized equation is $\dot{u} = A(t)u \quad (*)$ where $A(t+T) = A(t)$.

   (b) Let $\Phi(t)$ be the PSM for $*$, i.e., $\dot{\Phi} = A(t)\Phi, \Phi(0) = I$. Prove that $\Phi(t+T) = \Phi(t)\Phi(T)$.

   (c) Let B be a matrix such that $\Phi(T) = e^{BT}$. Show that $P(t)$ defined by $\Phi(t) = P(t)e^{Bt}$ is T-periodic, i.e., $P(t+T) = P(t)$.

   (d) Discuss the linearized stability of $x_p(t)$ in terms of the result in (c).

PDEs, Jan. 2000

Solve two out of the following three problems.

1) If (x, y) denote Cartesian coordinates in the plane, then the Laplace operator is, by definition,

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \; .$$

Formulate, as precisely as you can, what it means that the Laplace operator in polar coordinates (r, θ) reads

$$\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

and prove this statement.

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2) Consider the equation

$$u_t(x, t) = x u_x(x, t), \quad x \in \mathbb{R}, \quad t \geq 0 \; ,$$

with initial condition

$$u(x, 0) = \sin x \; .$$

a) Sketch the characteristics and solve the problem.

b) Solve the equation

$$u_t(x, t) = x u_x(x, t) + 1$$

with the same initial condition.

3) Consider the 1D wave equation

$$u_{tt}(x, t) = c^2 u_{xx}(x, t), \quad c > 0 \; ,$$

with initial condition

$$u(x, 0) = f(x), \quad u_t(x, 0) = g(x), \quad x \in \mathbb{R} \; ,$$

where f and g are real C^2 functions.

a) Solve the problem for

$$f(x) \equiv 1, \quad g(x) = 2\cos x, \quad c = 2 \; .$$

b) Assume that the solution $u(x, t)$ of the general problem is 2π–periodic in x. Show that the 'energy'

$$\|u_x(\cdot,t)\|^2 + \frac{1}{c^2}\|u_t(\cdot,t)\|^2$$

is constant in time. Here

$$\|v(\cdot)\|^2 = \int_0^{2\pi} v^2(x)dx \; .$$