ODE Exam
August 2000

Instructions: Answer problem 1 and either 2 or 3. Clearly indicate which of problems 2 or 3 you want graded.

1. Consider the nonlinear oscillator (NO) $\ddot{x} + g(x) = 0$ or in system for $\dot{x} = y, \dot{y} = -g(x)$.

   (a) Define a critical point (CP) (x_c, y_c) in terms of g.

   (b) Let (x_c, y_c) be a CP. Linearize the NO equation about the CP and discuss linearized stability in terms of g. What can you conclude at this stage about (nonlinear) stability?

   (c) Show that $E(x, y) = \frac{1}{2}y^2 + G(x)$ where $G'(x) = g(x)$ is constant along any solution of the NO equation.

   (d) Construct the phase plane portrait for the NO with $g(x) = x - x^3$. Discuss the qualitative behavior of solutions for all IC's (x_0, y_0). Include the CPs and their (nonlinear) stability in your discussion.

   (e) Show that on the unbounded part of the separatrix the solution blows up in finite time.

2. Consider $\dot{x} = A(t)x \quad (*)$, where $x \in \mathbf{R}^3$ and A is skew-symmetric, i.e., $A^T = -A$.

   (a) Let $\Phi(t)$ be the Principle Solution Matrix defined by $\dot{\Phi} = A(t)\Phi$, $\Phi(0) = I$. Let $\Gamma_1(t) := \Phi(t)\Phi(t)^T$ and $\Gamma_2(t) := \Phi(t)^T\Phi(t)$. Find differential equations for Γ_1 and Γ_2 and use uniqueness to show that $\Gamma_1(t) = I = \Gamma_2(t)$.

   (b) Let $T(t) = [u(t)v(t)w(t)]$ be an orthogonal matrix such that $w(t)$ is a solution of $*$. Define the transformation $\Phi \rightarrow \Psi$ via $\Phi(t) = T(t)\Psi(t)$. Show that $\dot{\Psi} = C(t)\Psi$ where $C(t)$ is skew-symmetric with zero third column.

   (c) Solve $\dot{\Psi} = C(t)\Psi$, $\Psi(0) = T^T(0)$, where $C(t) = c_{21}(t)\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Solve first for $c_{21} = constant$ and then generalize.

   (d) Explain how knowing one solution of $*$ leads to the general solution. Here you may assume $T(t)$ can be constructed as a smooth function given $w(t)$.

3. Consider the IVP $\dot{x} = f(x; a), x(0) = z$, and let $\varphi(t, a)$ be the solution with $\varphi(0, a) = z$.

   (a) Define continuity of φ in a for fixed t.

   (b) Specify general conditions on f so that solutions of the IVP exist uniquely on some open interval containing $t = 0$ and are continuous in a for fixed t.

   (c) Prove the continuity in a for fixed t for your conditions in (b). Use a form of the Gronwall inequality in your proof.

   (d) Prove the version of the Gronwall inequality you used in (c).

PDE Exam, Aug., 2000

Do 2 out of the following 3 problems.

1. Consider the following two PDEs for a function $u(x,t)$,
a)

$$u_t + xu_x = 0$$

and
b)

$$u_t + xu_x + 2u = 0 \ .$$

Solve the two PDEs with the initial condition

$$u(x,0) = \cos x, \quad x \in \mathbb{R} \ .$$

2a) The initial value problem

$$\begin{aligned} u_t &= u_{xx}, \quad x \in \mathbb{R}, \quad t \geq 0 \ , \\ u(x,0) &= \cos x, \quad x \in \mathbb{R} \ , \end{aligned}$$

is easily solved by separation of variables. Obtain the solution.

b) Now consider the problem

$$\begin{aligned} u_t &= u_{xxx}, \quad x \in \mathbb{R}, \quad t \geq 0 \ , \\ u(x,0) &= \cos x, \quad x \in \mathbb{R} \ . \end{aligned}$$

By writing $\cos x$ in terms of complex exponentials, one can again solve by separation of variables. Carry this out and obtain the solution in its real form.

3a) Derive d'Alembert's solution of the initial value problem

$$u_{tt} = c^2 u_{xx}, \quad x \in \mathbb{R}, \quad t \geq 0 \ ,$$
$$u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad x \in \mathbb{R} \ ,$$

where f and g are smooth functions and $c > 0$. Here you may use the ansatz $u(x,t) = U(x+ct) + V(x-ct)$.

b) Assume that $f(x)$ and $g(x)$ are odd functions. Show that $u(0,t) = 0$ for all $t \geq 0$.

c) Use b) to solve the initial–boundary value problem

$$u_{tt} = c^2 u_{xx}, \quad x \in \mathbb{R}, \quad t \geq 0 \ ,$$
$$u(x,0) = x^2, \quad u_t(x,0) = 0, \quad x \geq 0 \ ,$$
$$u(0,t) = 0, \quad t \geq 0 \ .$$

Is the solution a C^∞ function?