# ODE Preliminary Exam
**Summer 2001**

**Directions:** Do one of problems (1) and (2), and do both of problems (3) and (4).

**1.** Consider the linear ODE

$$\dot{\mathbf{x}} = A\mathbf{x} + B(t)\mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0.$$

Suppose that all of the eigenvalues of $A \in \mathbb{R}^{n\times n}$ have negative real part, and that $B(t) \in \mathbb{R}^{n\times n}$ is continuous and satisfies

$$\lim_{t\to\infty} \|B(t)\| = 0.$$

Show that $\mathbf{x} = \mathbf{0}$ is asymptotically stable.

**2.** Consider the linear system

$$\dot{\mathbf{x}} = A(t,\gamma)\mathbf{x}, \tag{1}$$

where $A(t,\gamma) \in \mathbb{C}^{n\times n}$ is continuous and uniformly bounded for all $t \in \mathbb{R}$. Let $\Phi(t,\gamma)$ be the fundamental matrix solution satisfying $\Phi(0,\gamma) = I_n$, where $I_n \in \mathbb{R}^{n\times n}$ is the identity matrix.

a. Show that $\Phi(t,\gamma)^{-1}$ exists for all $t \in \mathbb{R}$.

b. Show that $\Psi(t,\gamma) = \Phi(t,\gamma)^{-1}$ satisfies the linear ODE $\dot{\Psi} = -\Psi A(t,\gamma)$.

c. Suppose that $A(t,\gamma) + A(t,\gamma^*)^\dagger = 0$, where "$\dagger$" represents the Hermitian of a matrix. Show that $\Psi(t,\gamma)^\dagger = \Phi(t,\gamma^*)$.

**3.** Consider the system

$$\begin{aligned} \dot{x} &= \phi(x,y)x + \psi(x,y)y \\ \dot{y} &= -\psi(x,y)x + \phi(x,y)y, \end{aligned} \tag{2}$$

where $\phi$, $\psi : \mathbb{R}^2 \to \mathbb{R}$ are smooth. Assume that $\psi(x,y) \neq 0$ for $(x,y) \neq (0,0)$. Furthermore, assume that there exists $0 < A < B$ such that $\phi(x,y) > 0$ for $x^2 + y^2 < A$, while $\phi(x,y) < 0$ for $x^2 + y^2 > B$. Show that there exists at least one nontrivial periodic orbit to equation (2). (Hint: Use polar coordinates).

**4.** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be smooth and satisfy $f(t+T,x) = f(t,x)$ for some $T > 0$ and all $(t,x) \in \mathbb{R}^2$. Consider the ODE

$$\dot{x} = f(t,x), \quad x(0) = x_0.$$

a. If $\phi(t)$ is a solution for all $t \geq 0$, show that each function $\phi_k(t) = \phi(t+kT)$ is also a solution for all $t \geq 0$.

b. Show that the ODE has a periodic solution if and only if it has a bounded solution. (Hint: If $\phi(t)$ is a bounded solution, consider $\phi_k(t)$ in each of the cases $\phi(0) = \phi(T)$, $\phi(0) < \phi(T)$, $\phi(0) > \phi(T)$.)

# PDE Preliminary Exam
**Summer 2001**

**Directions:** Work 3 of the 4 problems. Clearly indicate the 3 problems you want graded.

1. Consider the nonhomogeneous initial boundary problem:

$$\begin{aligned} \partial_t u - \partial_{xx} u &= q(x,t), \\ u(x,0) &= f(x), \\ u(0,t) = u(1,t) &= 0. \end{aligned} \tag{3}$$

   (a) Find the complete set of eigenfunctions, $(\phi_n(x))_1^\infty$ associated with this problem.

   (b) Assuming that $q$ and $f$ can be expanded in the above eigenbasis, solve the problem.

2. One dimensional wave equation.

   (a) Consider the IVP

$$\begin{aligned} \partial_{tt} u - \partial_{xx} u &= 0, \\ u(x,0) = g(x), \partial_t u(x,0) &= h(x). \end{aligned} \tag{4}$$

   Argue that $u(x,t) = F(x-t) + G(x+t)$ and use it to derive d'Alembert's formula $2u(x,t) = g(x+t) + g(x-t) + \int_{x-t}^{x+t} h(y)dy$.

   (b) Consider the nonhomogeneous problem

$$\begin{aligned} \partial_{tt} u - \partial_{xx} u &= f(x,t), \\ u(x,0) = 0, \partial_t u(x,0) &= 0. \end{aligned} \tag{5}$$

   Let $v = v(x,t;s)$ be defined by $\partial_{tt} v - \partial_{xx} v = 0$, with $v(x,s;s) = 0$ and $\partial_t v(x,s;s) = f(x,s)$. Use a) to find $v$ and show that $u(x,t) = \int_0^t v(x,t;s)ds$ solves the nonhomogeneous problem.

3. Consider the IVP

$$\begin{aligned} \partial_t u - u\partial_x u &= 0, \\ u(x,0) &= \frac{1}{\sqrt{2\pi}} \exp(-x^2/2). \end{aligned} \tag{6}$$

   (a) Find the characteristics and sketch them.

   (b) Find the breaking time, $t_b$.

   (c) Sketch the solution for $t = 0, 0 < t < t_b$, and $t = t_b$.

4. Poisson's Equation

   (a) Consider $\Delta u = 0$ in $\mathbf{R}^n$. Let $u(x) = v(r)$ where $r = (x_1^2 + \cdots + x_n^2)^{1/2}$. Find the DE for $v$ and solve it for $n = 2$.

   (b) Show that $\Phi(x) = -\frac{1}{2\pi} \log |x|$ satisfies $\Delta u = 0$ in $\mathbf{R}^2 \backslash \{(0,0)\}$. Note that here $x = (x^1, x^2)$ and $|x| = r$.

   (c) Consider $u(x) = \int_{\mathbf{R}^2} \Phi(x-y) f(y) dy$ for $x \in \mathbf{R}^2$.

      i. What's wrong with the argument $\Delta u = \int_{\mathbf{R}^2} \Delta_x \Phi(x-y) f(y) dy = 0$?

      ii. Show that $\Delta u = f(x)$. Hint: Write the convolution in reverse order and break up $\mathbf{R}^2$ into $|y| < \epsilon$ and its complement.