ODE Preliminary Exam

Winter 2002

Directions: Do three out of the following four problems.

1. Consider the ODE

$$\dot{\mathbf{x}} = A\mathbf{x} + f(t, \mathbf{x}), \quad \mathbf{x}(0) = \mathbf{x}_0.$$

Suppose that all of the eigenvalues of $A \in \mathbb{R}^{n\times n}$ have negative real part, that $f(t, \mathbf{x})$ is continuous, and that for any $\epsilon > 0$ there is a $\delta > 0$ such that

$$|f(t, \mathbf{x})| \leq \epsilon|\mathbf{x}|, \qquad |\mathbf{x}| < \delta, \; t \in \mathbb{R}.$$

Show that $\mathbf{x} = \mathbf{0}$ is uniformly asymptotically stable.

2. Consider the second-order equation

$$\ddot{x} + p(t)x = 0, \quad p(t) = -2\,\frac{1+\cos t}{2-\sin t}. \tag{1}$$

a. Verify that

$$x_1(t) = (2 - \sin t)e^{-t}$$

is a solution.

b. Are all solutions to equation (1) bounded? Explain. (Hint: Note that $p(t + 2\pi) = p(t)$.)

3. Consider the system

$$\begin{aligned} \dot{x} &= -x + \tfrac{1}{4}ay + x^2 y \\ \dot{y} &= \tfrac{1}{2} - \tfrac{1}{4}ay - x^2 y. \end{aligned} \tag{2}$$

where $0 < a < 2\sqrt{3} - 3$. Show that there is at least one nonconstant periodic solution to the system.

4. Consider the initial value problem

$$\dot{x} = f(t, x), \quad x(0) = x_0$$

where $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is smooth and satisfies

$$|f(t, x)| \leq \phi(t)|x|, \qquad \int_0^{+\infty} \phi(t)\,dt < \infty \quad (\phi(t) \geq 0).$$

a. Show that a solution exists for all $t \geq 0$.

b. Show that every solution approaches a constant as $t \to \infty$.

PDE Preliminary Exam
Winter 2002

Directions: Do all three of the following problems.

5. Solve

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},$$

on the interval $x \in [-1, 1]$ subject to the boundary conditions

$$\frac{\partial u}{\partial x}(x = \pm 1, t) = 0,$$

and the initial condition

$$u(x, t = 0) = \begin{cases} -1, & -1 < x < 0 \\ 1, & 0 < x < 1. \end{cases}$$

6. Solve Laplace's equation $\Delta u = 0$ in a disk $\sqrt{x^2 + y^2} \leq 1$ with Neumann boundary conditions

$$\frac{\partial u}{\partial r}\,|_{r=1} = f(\phi),$$

where

$$f(\phi) = \begin{cases} -\pi + 2\phi, & 0 \leq \phi \leq \pi \\ 3\pi - 2\phi, & \pi \leq \phi \leq 2\pi. \end{cases}$$

7. Find the general solution of the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2},$$

on the interval $x \in [0, +\infty)$ with boundary conditions

$$\frac{\partial u}{\partial x}\,|_{x=0} = 0.$$

Find the particular solution for the given initial conditions:

$$u(x, t = 0) = \phi(x), \quad \frac{\partial u}{\partial t}(x, t = 0) = \psi(x).$$