DE Preliminary Exam

January 2004

Directions: Do all six of the following problems. Show all of your work, and justify all of your calculations.

1. Consider the ODE

$$\dot{x} = v \times x,$$

where $\times$ denotes the cross-product in $\mathbb{R}^3$. For each of the following scenarios, discuss the stability of $x = 0$:

(a) $v \in \mathbb{R}^3$ is constant

(b) $v = v(t) \in \mathbb{R}^3$, where $v(t + 2\pi) = v(t)$ for all $t \in \mathbb{R}$.

2. Consider the initial value problem

$$\dot{x} = (A + B(t))x, \quad x(0) = x_0,$$

where

$$A := \begin{pmatrix} -3 & -2 \\ 2 & -3 \end{pmatrix}, \quad B(t) := \begin{pmatrix} \mathrm{e}^{-t} & \operatorname{sech}(2t) \\ \frac{1}{1+t^2} & \mathrm{e}^{-2t} \end{pmatrix}.$$

Show that $x(t) \to 0$ as $t \to +\infty$ for each initial value x_0.

3. Consider the ODE

$$\begin{aligned} \dot{x} &= x - y - x^3 \\ \dot{y} &= x + y - y^3. \end{aligned}$$

Show that:

(a) there is at least one nontrivial periodic solution

(b) the periodic solution found in part (a) is unique and globally attracting.

4. Consider the IVP

$$u_t - (xu)_x = 0; \quad u(x, 0) = u_0(x),$$

where $(x, t) \in \mathbb{R} \times \mathbb{R}^+$.

(a) Find the characteristics through $(x, t) = (x_0, 0)$ and sketch them.

(b) Discuss the behavior of the solution along the characteristic through $(x, t) = (x_0, 0)$.

(c) Find u(x,t).

5. Consider the IBVP

$$u_t - u_{xx} = 0; \quad u(x,0) = f(x), \quad u(0,t) = u(\pi,t) = 0.$$

where $(x,t) \in (0,\pi) \times \mathbb{R}^+$.

(a) Find a candidate for the solution of the IBVP of the form

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t)\phi_n(x), \tag{0.1}$$

where $\{\phi_n(x)\}_{n=1}^{\infty}$ is a complete set of eigenfunctions.

(b) Give sufficient conditions on f such that

$$\lim_{N\to\infty} \sum_{n=1}^{N} f_n\phi_n(x) = f(x),$$

where f_n is appropriately defined for each n.

(c) Outline a proof that equation (0.1) actually solves the IBVP using uniform convergence arguments. State any and all assumptions on f that you require.

(d) State a maximum principle for the IBVP, and show that $b_1(t)\phi_1(x)$ satisfies it.

6. Solve the IVP for the nonhomogeneous Klein-Gordon equation (KGE),

$$E_{xx} - \kappa^2 E - E_{tt} = S(x,t); \quad E(x,0) = E_t(x,0) = 0,$$

as follows:

(a) Apply Duhamel's principle; that is, find the IVP satisfied by $e(x,t;s)$ so that $E(x,t) = \int_0^t e(x,t;s)$. (Hint: e should satisfy a homogeneous 1D KGE.)

(b) Define $f(x,y,t;s) := \mathrm{e}^{\mathrm{i}\kappa y} e(x,t;s)$, and show that f satisfies a homogeneous 2D wave equation. Solve for f using the fact that the solution $u(x,t)$, $x \in \mathbb{R}^2$, of the IVP

$$u_{tt} = \Delta u; \quad u(x,0) = h(x), \quad u_t(x,0) = g(x)$$

is

$$u(x,t) = \frac{1}{2\pi}\frac{\mathrm{d}}{\mathrm{d}t}\iint_{|\xi|<t} \frac{h(x+\xi)}{t^2+|\xi|^2}\,\mathrm{d}\xi + \frac{1}{2\pi}\iint_{|\xi|<t} \frac{g(x+\xi)}{\sqrt{t^2+|\xi|^2}}\,\mathrm{d}\xi.$$

(c) Show that

$$e(x,t;s) = -\frac{1}{2}\int_{-(t-s)}^{t-s} J(\kappa\sqrt{(t-s)^2-\xi_1^2}\,)S(x+\xi_1,s)\,\mathrm{d}\xi_1,$$

where

$$J(x) := \int_{-1}^{1} \frac{\mathrm{e}^{\mathrm{i}xs}}{\sqrt{1-s^2}}\,\mathrm{d}s.$$

(d) Find $u(x,t)$.