Original

ODE/PDE qualifying exam, August 2004

Complete all eight problems. Write the last four digits of your SSN (*not* your name) on each sheet. Clear and concise answers with good justification will improve with your score.

1. Consider $\dot{x} = Ax$, $A = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$.

   (a) Find the eigen solutions and the general solution.

   (b) Use the result of a) to construct $\exp At$.

   (c) Sketch the phase plane portrait and discuss the stability.

2. Let $H : \mathbf{R}^2 \to \mathbf{R}$ via $(x, y) \mapsto H(x, y)$ be given. Consider the system

$$\dot{x} = \frac{\partial}{\partial y} H(x, y), \quad \dot{x} = -\frac{\partial}{\partial x} H(x, y). \tag{1}$$

   (a) Let $(x(t), y(t))$ be a solution. Show that $h(t) = H(x(t), y(t))$ is constant.

   (b) Let $H(x, y) = \frac{1}{2}y^2 + G(x)$, $G(x) = \frac{1}{2}x^2 - \frac{1}{40}x^5$.

      i. Find the equilibrium.

      ii. Linearize about the equilibrium and discuss linear stability. What can you conclude about nonlinear stability (state the standard thm.).

      iii. Sketch the phase plane portrait (level curves of H) and complete the discussion of nonlinear stability.

      iv. What initial conditions give rise to periodic solutions. Discuss their stability in terms of Lyapunov and orbital stability.

3. Consider $\dot{x} = f(x)$ and $\dot{y} = f(y) + g(y)$, (the latter can be viewed as a perturbation of the former) where f is globally Lipschitz and g is bounded. Find a bound on $|x(t) - y(t)|$ in the case where $x(0) = a$ and $y(0) = b$.

4. Suppose $x_p(t)$ is a T-periodic solution of $\dot{x} = f(x, t), x \in \mathbf{R}^n$, where $f(x, t) = f(x, t+T)$ is smooth.

   (a) Let x be a solution of the ODE and then define u by $x = x_p + u$. Find and then solve the linearized equation for u, i.e., find $A(t)$ in terms of f such that the linearized equation is $\dot{u} = A(t)u$ $(*)$ where $A(t + T) = A(t)$.

   (b) Let $\Phi(t)$ be the PSM for $*$, i.e., $\dot{\Phi} = A(t)\Phi, \Phi(0) = I$. Prove that $\Phi(t + T) = \Phi(t)\Phi(T)$.

   (c) Let B be a matrix such that $\Phi(T) = e^{BT}$. Show that $P(t)$ defined by $\Phi(t) = P(t)e^{Bt}$ is T-periodic, i.e., $P(t + T) = P(t)$.

(d) Outline the proof that the periodic solution in a) is asymptotically stable if the eigenvalues of B in c) have negative real parts.

5. Solve

$$x\,u_x + (x+y)\,u_y = u+1\,, \quad u(x,0) = x^2\,, \tag{2}$$

where $u = u(x,y)$ and $u_x = \partial u/\partial x$.

6. This problem concerns the wave equation.

(a) Use separation of variables to find $u = u(x,t)$, $0 \le x \le 1$, $t \ge 0$, where

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}\,, \tag{3}$$

$$u(0,t) = 0\,, \quad u(1,t) = 0\,, \tag{4}$$

and

$$u(x,0) = f(x)\,, \quad \frac{\partial u}{\partial x}(x,0) = g(x)\,, \tag{5}$$

where f and g are two given smooth functions.

(b) Is the problem in part (a) well posed?

(c) Does the solution of the problem in part (a) satisfy a maximum principal?

7. Let Ω be a bounded domain in two dimensions with smooth boundary. The Dirichlet boundary-value problem for the Laplacian in Ω is to find $u = u(x,y)$ satisfying $\Delta u = 0$ in Ω and satisfying the boundary condition $u|_{\partial\Omega} = g$ for some given smooth g. For the following, assume (as is well known) that this problem has at least one solution. Do not assume the solution is unique or depends continuously on the data of the problem.

(a) State a maximum principle for this boundary-value problem.

(b) Show that the maximum principle implies that the solutions of this boundary-value problem are unique.

(c) Introduce the two norms

$$\|u\|_\Omega = \max_{(x,y)\in\Omega} |u(x,y)|\,, \tag{6}$$

$$\|u\|_{\partial\Omega} = \max_{(x,y)\in\partial\Omega} |f(x,y)|\,. \tag{7}$$

State the maximum principle in terms of these norms (if you have not done this in part (a)).

(d) What does it mean that the solution of the boundary-value problem depends continuously on the data? Is the boundary-value problem well posed?

8. Let H be the space of smooth complex-valued functions on the interval $0 \leq x \leq 1$ with the inner product

$$\langle f, g \rangle = \int_0^1 f(x)\,\bar{g}(x)\,dx\,, \quad f, g \in H\,. \tag{8}$$

Also let A be an operator with domain $\mathcal{D}(A)$:

$$A = \frac{d^2}{dx^2}\,, \quad \mathcal{D}(A) = \{f \in H, f'(0) = f'(1) = 0\}\,, \tag{9}$$

that is, f satisfies Neumann boundary conditions. This operator has eigenvalues and orthonormal eigenfunctions given by:

$$\lambda_n = -n^2\pi^2\,, \quad \phi_n(x) = \sqrt{2}\cos(n\pi x)\,, \quad n \geq 0\,. \tag{10}$$

In this problem, be careful about $n = 0$. Use this information to answer the following:

(a) Solve the problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}\,, \tag{11}$$

$$\frac{\partial u}{\partial x}(0,t) = 0\,, \quad \frac{\partial u}{\partial x}(1,t) = 0\,, \tag{12}$$

and

$$u(x,0) = f(x)\,, \tag{13}$$

where f is a given smooth function.

(b) In part (a) what is the $\lim_{t\to\infty} u(x,t)$?

(c) In part (a) does the solution satisfy a maximum principle?

(d) Give the Green's function for the problem in part (a).

(e) Use part (a) to solve

$$\frac{\partial^2 u}{\partial x^2} = f \tag{14}$$

$$\frac{\partial u}{\partial x}(0,t) = 0\,, \quad \frac{\partial u}{\partial x}(1,t) = 0\,, \tag{15}$$

where f is a given smooth function.

(f) Is the problem in part (e) well posed?