ODE/PDE Qualifying Examination. January 2005

Directions:

Please answer all six questions. Start each question on a new sheet of paper. Write only on one side of each sheet of paper. Number the pages in the order you want them to be read, and identify yourself in each page by writing *only* the last four digits of your SS #. **Please do not write your name in the exam**.

Questions:

1. Consider the initial value problem (IVP)

$$\begin{aligned} &y' = -y^2 + (\cos t)y, \quad 0 < t < \infty \quad y \in \mathbf{R} \\ &y(0) = y_0 \end{aligned} \tag{1}$$

   (a) Find all constant solutions $y = y(t)$ of (1).

   (b) Prove that if $y_0 > 0$ then $y(t) > 0$ for $t > 0$.

   (c) Prove that if $y = y(t)$ is a solution of (1) with $y_0 > 0$, then $y = y(t)$ remains bounded for all times (you can assume that the solution exists on $0 \le t < \infty$.)

2. Consider the nonlinear second-order equation

$$x'' + 3x' + 2xe^x = 0 \tag{2}$$

   (a) Introduce the additional variable $v = x'$ and recast (2) as a 2×2 ODE system.

   (b) Show that this 2×2 system has only one equilibrium point. Linearize the system around this equilibrium point.

   (c) Find the general solution of the linearized system and discuss the nonlinear stability of the equilibrium point. Sketch the phase portrait near the equilibrium point.

3. (a) Show that if there exists a function $\mu = \mu(x, y)$ such that $\dfrac{\partial}{\partial x}(\mu f) + \dfrac{\partial}{\partial y}(\mu g) \neq 0$ for all (x, y) in a simply connected region R, then the planar autonomous system

$$\begin{aligned} x' &= f(x, y) \\ y' &= g(x, y) \end{aligned}$$

   has no periodic orbits in R.

   (b) Now consider the ODE system

$$\begin{aligned} x' &= (1 + x + y)x \\ y' &= (2 + x + 3y)y \end{aligned} \tag{3}$$

   Show that the x-axis and the y-axis consists of orbits.

   (c) Show that any solution of (3) starting in the first quadrant $x > 0$, $y > 0$ stays in the first quadrant.

   (d) Show that there are no periodic orbits in the first quadrant (use $\mu(x, y) = 1/xy$).

4. Solve the initial-boundary-value problem

$$\begin{aligned}
&u_t = u_{xx} + 9\cos 3x \quad 0 < x < \pi, \quad 0 < t < \infty \\
&u_x(0,t) = 0 \\
&u_x(\pi,t) = 0 \\
&u(x,0) = 1 + \cos x,
\end{aligned}$$

and discuss the behavior of $u(x,t)$ as $t \to \infty$.

Note: In case you need it, $\displaystyle\int_0^\pi \cos^2 nx\, dx = \frac{\pi}{2}$, $n = 1, 2, 3, ...$

5. (a) Solve the initial value problem

$$\begin{aligned}
&\frac{\partial u}{\partial t} + x\frac{\partial u}{\partial x} = -u^2, \quad -\infty < x < \infty, \quad 0 < t < \infty \\
&u(x,0) = f(x),
\end{aligned}$$

where $f(x) \geq 0$. Discuss the behavior of $u(x,t)$ as $t \to \infty$.

(b) If $f(x) > 0$ on $0 < x < 1$ and $f(x) = 0$ elsewhere, plot the region in the (x,t)-plane where $u(x,t) > 0$.

6. (a) Prove that $E(x,y) = \dfrac{1}{2\pi}\ln r$, where $r = \sqrt{x^2+y^2}$, satisfies $\Delta E(x,y) = \delta(x,y)$ in distribution sense. Please show all the details.

Hint: In polar coordinates $\Delta = \dfrac{1}{r}\dfrac{\partial}{\partial r}r\dfrac{\partial}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial}{\partial \theta^2} = \dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2}{\partial \theta^2}$.

(b) If $f = f(x,y)$ is a smooth function of bounded support in $\mathbf{R}^2$, what is the solution of $\Delta u = f$?