ODE/PDE, Fall 2007

MS/PhD Qualifying Examination

1a) Let $M \in \mathbb{C}^{n\times n}$ be a constant matrix. State (without proof) necessary and sufficient conditions in terms of the eigenvalues of M which guarantee that

(i) $w(t) \to 0$ as $t \to \infty$ for every solution $w(t)$ of the system $w' = Mw$;

(ii) every solution $w(t)$ of the system $w' = Mw$ is bounded for $t \in \mathbb{R}$.

b) Let $B \in \mathbb{C}^{n\times n}$ be a constant matrix and consider the second–order system $u'' = Bu$. Is it possible that $u(t) \to 0$ as $t \to \infty$ for every solution $u(t)$? Under what assumptions on B is every solution $u(t)$ bounded for $t \in \mathbb{R}$? Justify your answers.

2) Consider the following two ordinary boundary value problems:

$$u''(x) + \pi^2 u(x) = 1, \quad u(0) = u(1) = 0 \ ,$$

and

$$u''(x) + \pi^2 u(x) = \sin(2\pi x), \quad u(0) = u(1) = 0 \ .$$

One of the problems has no solution, the other has infinitely many solutions. Solve the one that has infinitely many solutions and explain. Hint: Consider the corresponding homogeneous problem $v'' + \pi^2 v = 0, v(0) = v(1) = 0$.

3) The beam equation is given by:

$$\frac{\partial^2 w}{\partial t^2} + \gamma^2 \Delta^2 w = 0 \ ,$$

where Δ is the Laplacian.

a) Use Fourier transforms to formally solve the Cauchy problem for the beam equation. The Cauchy data is given by $w(x,0) = f(x)$, $\frac{\partial w}{\partial t}(x,0) = g(x)$, $x \in R^d$.

b) Use Parseval's relation and the solution computed above to derive bounds on $\int_{R^d} w^2(x,t)dx$ anf $\int_{R^d} \left(\frac{\partial w}{\partial t}(x,t)\right)^2 dx$ in terms of f and g.

c) Now consider a smooth, bounded spatial domain, $x \in \Omega \subset R^d$ with boundary conditions $w = 0$ and $\frac{\partial w}{\partial \nu} \equiv \mathbf{n} \cdot \nabla u = 0$. (Here $\mathbf{n}$ is the outward unit normal to Ω.) Define:

$$\mathcal{E}(t) = \frac{1}{2} \int_\Omega \left(\left(\frac{\partial w}{\partial t}(x,t) \right)^2 + \gamma^2 \left(\Delta w(x,t) \right)^2 \right) dx \ .$$

Prove that $\mathcal{E}$ is constant.

d) Use the result above to prove that solutions to the initial-boundary value problem of part (c) are unique.

4) Consider the nonlinear conservation law:

$$\frac{\partial u}{\partial t} + \frac{\partial F(u)}{\partial x} = 0$$

where $u(x,t) \in R^n$ and $F(u(x,t)) \in R^n$, $(x,t) \in R \times (0,\infty)$.

a) Define $U(t; x_1, x_2) = \int_{x_1}^{x_2} u(x,t)dx$. Derive a formula for $\frac{dU}{dt}$ involving only the fluxes, F, at x_1 and x_2.

b) A weak solution of the conservation law satisfies:

$$\int_0^\infty \int_R \left(u(x,t)\frac{\partial v}{\partial t}(x,t) + F(u(x,t))\frac{\partial v}{\partial x}(x,t) \right) dxdt = 0$$

for all infinitely differentiable, compactly supported scalar functions $v(x,t)$. Consider the function:

$$u(x,t) = \begin{cases} u_L, & x < S(t), \\ u_R, & x > S(t). \end{cases}$$

Derive conditions on $\frac{dS}{dt}$ so that u is a weak solution. (The solution is called a shock wave and $\frac{dS}{dt}$ is the shock speed.)