ODE/PDE Qualifying Examination. January 2008

Directions:

Please answer all **five** questions. Start each question on a new sheet of paper. Write only on one side of each sheet of paper. Number the pages in the order you want them to be read, and identify yourself in each page by writing *only* the last four digits of your SS #. **Please do not write your name in the exam**.

Questions:

1. Consider the initial-value-problem (IVP) $\dfrac{dy}{dt} = -y^3$, $y(0) = a \in \mathbb{R}$, where $t \geq 0$.

   (a) Sketch the direction field and phase line, and discuss the stability of the equilibria.
   (b) Find the solution $y(t)$ and the maximal interval, (α, β), of existence as a function of a. Note: $\alpha = -\infty$ and $\beta = \infty$ are possible values.
   (c) Is the solution of this IVP unique. Why?
   (d) Denote the solution $y(t)$ of the IVP by $y(t) = \varphi(t, a)$. Show that $\varphi(t + \tau, a) = \varphi(t, \varphi(\tau, a))$ whenever they are defined.

2. Consider the initial-value-problem (IVP)

$$\frac{dx}{dt} = \epsilon f(x, t), \quad x(0) = x_0, \quad 0 < \epsilon \leq 1, \tag{1}$$

   and its perturbation

$$\frac{dy}{dt} = \epsilon f(y, t) + \epsilon^2 h(t, \epsilon), \quad y(0) = x_0, \quad 0 < \epsilon \leq 1. \tag{2}$$

   Assume that $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is smooth, bounded and globally Lipschitz ($|f(x,t) - f(y,t)| \leq L|x - y|$ for all $x, y \in \mathbb{R}^n$ and all $t \in \mathbb{R}$), and $h(t, \epsilon)$ is bounded for all $t \in \mathbb{R}$ and for $0 < \epsilon \leq 1$.

   (a) Assume that the solutions of (1) and (2) exist on $[0, T/\epsilon]$. Use Gronwall's inequality to show that $|x(t) - y(t)| \leq C(T)\epsilon$ for $0 \leq t \leq T/\epsilon$.
   (b) Assume now that $f(x, t)$ is 2π-periodic in t. Define $\bar{f}(x) = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} f(x, t)dt$, and consider the IVP

$$\frac{dv}{dt} = \epsilon \bar{f}(v), \quad v(0) = x_0. \tag{3}$$

   Consider a transformation of the form

$$y = v + \epsilon P(v, t), \tag{4}$$

   with $P(v, 0) = 0$ and $P(v, t + 2\pi) = P(v, t)$. Determine $P(v, t)$, independent of ϵ, so that $y(t)$ satisfies an equation of the form (2) (given that $v(t)$ satisfies (3)) and identify $h(t, \epsilon)$. Conclude that $v(t)$ is a good approximation of $x(t)$, i.e. argue that $|x(t) - v(t)| \leq C_1(T)\epsilon$ for $0 \leq t \leq T/\epsilon$.
   Hint for part (b): Start by differentiating $y(t)$ in (4).

3. Consider the first-order PDE $yu_x - g(x)u_y = 0$ with the boundary condition $u(0, y) = b(y)$ and $g(x) = x + x^3$. Solve for $u(x, y)$ using the method of characteristics and in the process find conditions on b so that the problem is well posed. Your solution should include:
(a) The characteristic equations, an associated conservation law and a sketch of the associated phase plane portrait.
(b) An interpretation of the solution in terms of the phase plane portrait.
(c) Argue that the solution can be written $u(x, y) = b(\sqrt{y^2 + 2G(x)})$ where $G(x) = \int_0^x g(t)dt$. This can be proven directly. The point here is to obtain this form from your method of characteristics.

4. Consider the initial-value-problem (IVP)

$$\frac{\partial u}{\partial t} = P\left(\frac{\partial}{\partial x}\right) u, \quad x \in \mathbb{R}, \quad 0 < t < \infty,$$
$$u(x, 0) = f(x).$$

Here $P\left(\frac{\partial}{\partial x}\right) = \sum_{j=0}^{R} a_j \frac{\partial^j}{\partial x^j}$, with $a_j \in \mathbb{C}$, $j = 0, ..., R$, and $f(x) \in L^2(\mathbb{R})$. We say that this IVP is well-posed in $C([0, T], L^2(\mathbb{R}))$ if and only if there is a constant $C(T) \geq 0$ such that the solution $u(x, t)$ satisfies

$$\max_{0 \leq t \leq T} \|u(\cdot, t)\|_{L^2} \leq C(T)\|f\|_{L^2}.$$

(a) Use the Fourier transform to show that if $\max_{\substack{0 \leq t \leq T \\ -\infty < k < \infty}} \left|e^{tP(2\pi ik)}\right| < \infty$, where $P(2\pi ik) = \sum_{j=0}^{R} a_j(2\pi ik)^j$, then the IVP is well-posed in $C([0, T], L^2(\mathbb{R}))$.
(b) Consider the special case where $u_t = u_{xxx} + Du_{xx} + u_x$. For which values of $D \in \mathbb{R}$ is the IVP well-posed in $C([0, T], L^2(\mathbb{R}))$?

5. (a) Given two distributions $u(x)$ and $f(x)$, $x \in \mathbb{R}^3$, define what it means for $\nabla^2 u = f$ in the distribution sense.
(b) Show that if $u(x) = -\frac{1}{4\pi r}$, with $r = |x| = \sqrt{x_1^2 + x_2^2 + x_3^2}$, and $f(x) = \delta(x)$, the Dirac delta, then $\nabla^2 u = \delta(x)$ in distribution sense.